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On the dyon partition function in $\mathcal{N} = 2$ theories

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Abstract: We study the entropy function of two $\mathcal{N} = 2$ string compactifications obtained as freely acting orbifolds of $\mathcal{N} = 4$ theories: the STU model and the FHSV model. The Gauss-Bonnet term for these compactifications is known precisely. We apply the entropy function formalism including the contribution of this four derivative term and evaluate the entropy of dyons to the first subleading order in charges for these models. We then propose a partition function involving the product of three Siegel modular forms of weight zero which reproduces the degeneracy of dyonic black holes in the STU model to the first subleading order in charges. The proposal is invariant under all the duality symmetries of the STU model. For the FHSV model we write down an approximate partition function involving a Siegel modular form of weight four which captures the entropy of dyons in the FHSV model in the limit when electric charges are much larger than magnetic charges.

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1. Introduction

Recent studies has led to a good understanding of the spectrum of 1/4 BPS dyonic states in a class of $\mathcal{N} = 4$ supersymmetric string theories [1 − 13]. These theories are a class of generic $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifold of type IIA string theory on $K3 \times T^2$ or T^6. In each example studied so far, the statistical entropy computed by taking the logarithm of the degeneracy of states agrees with the entropy of the corresponding black hole for large charges. This is not only in the leading order, but also in the first sub-leading order [2, 6, 9 − 11]. On the black hole side this requires taking into account the effect of the Gauss-Bonnet term in the low energy effective action of the theory, and the use of the Wald's generalized formula for the black hole entropy in the presence of higher derivative corrections [14 − 16]. These 1/4 BPS dyons are known to have regions of marginal stability, the degeneracies of the dyons jump across these regions of marginal stability. These changes are precisely captured by the same dyon partition function but with different choices of the integration contour, the moduli dependence of the contour of integration is known precisely [17–23]. For a recent review on these topics see [24].

So far similar studies in $\mathcal{N} = 2$ theories has been lacking. Because of reduced supersymmetry the Gauss-Bonnet term in the low energy effective action is not just a term dependent on the axion and dilaton, but also on the other moduli of the vector multiplets. The precise agreement of the asymptotic degeneracy of the dyons in $\mathcal{N} = 4$ theory to the first sub-leading order depended crucially on just the axion-dilaton dependence of the Gauss-Bonnet term. Any similar proposal for dyons in $\mathcal{N} = 2$ theories should address the question the moduli dependence of the other vector multiplets.

The study of 1/2 BPS dyons in a generic $\mathcal{N} = 2$ theory would be a hard task. In this paper we focus on two $\mathcal{N} = 2$ theories which are closely related to $\mathcal{N} = 4$ theories. The first theory is the STU model obtained by a freely acting orbifold action of type IIA on $T^4 \times T^2$. There are three vector multiplets in this theory, the S, T and the U. This model was constructed and studied in [25, 26], the coefficient of the Gauss-Bonnet term in this model is known exactly and it is a sum of contributions from the S, T and the U-moduli. In this case the exact moduli space of the vector multiplets is given by

$$\mathcal{M}_V = \left.\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right|_S \times \left.\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right|_T \times \left.\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right|_U . \tag{1.1}$$

where each $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ factor parameterizes the respective moduli. We study the entropy function for this model and evaluate the expectation values of the moduli at the attractor point. This enables us to extract the S, T, U duality invariants constructed out of bilinears of the electric and magnetic charges which characterize the entropy at the next leading order. Using these invariants we propose a partition function for dyons in this model. This partition function involves the product of three Siegel modular forms of weight zero, it has all the duality symmetries of the STU model. The statistical entropy obtained from the partition function reproduces correctly the entropy of a dyonic black hole to the first sub-leading order for large values of charges.[1]

Another well known example of a $\mathcal{N} = 2$ model closely related to a parent $\mathcal{N} = 4$ theory is that of the self mirror Calabi-Yau constructed in [27]. On the type IIA side it is constructed by a freely acting orbifold of type IIA on $K3 \times T^2$, the resulting Calabi-Yau is a self-mirror manifold. Therefore the moduli space of this theory is known exactly. The vector multiplet moduli space is given by

$$\mathcal{M}_v = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,10)}{\mathrm{SO}(2)\times\mathrm{SO}(10)}. \tag{1.2}$$

One of the important property of this moduli space is that the dilaton-axion moduli factorizes from the rest of the vector multiplet moduli. The dilaton-axion parameterizes the coset $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ while the rest of the moduli(the T-moduli) parameterize the coset $\mathrm{SO}(2,10)/\,\mathrm{SO}(2)\times\mathrm{SO}(10)$. Because of this factorization, the coefficient of the Gauss-Bonnet term can be computed exactly [28]. From the analysis of the entropy function for

[1] For the reader who is interested in the formula for the partition function, please see (4.3).

the FHSV model and the attractor values of the T-moduli we show that this factorization allows one to parametrically separate the subleading contribution to the entropy of dyons in this model into two parts. The contribution from the axion-dilaton dependence of the Gauss-Bonnet term is dominant when the electric charges are much larger than the magnetic charges. We then write down a Siegel modular form of weight 4 which captures the dependence of the entropy function on the axion-dilaton moduli.

The organization of the paper is as follows: In section 2. we review the construction of the STU model as well as the FHSV model and recall the the coefficient of the Gauss-Bonnet term in these models. Section 3. contains the entropy function analysis of these models. We explictly solve for the attractor moduli in both these models. This enables us to evaluate the sub-leading contribution to the black hole entropy of dyons from the coefficient of the Gauss-Bonnet term in these models. It also helps us to determine the charge bilinears characterizing the entropy at the subleading order. In section 4. we propose a partition function for dyons in the STU model and show that it has the required duality invariance and reproduces correctly the entropy of a dyonic black hole to the first sub-leading order for large values of the charges. In section 5. we write down an approximate partition function for dyon in the FHSV model which captures the entropy of the corresponding black hole for large values of the charges but with electric charges much larger than the magnetic charges. The appendices contain conventions regarding the 't Hooft symbols for SO(2,2) and the properties of the Siegel modular forms used in the paper. Appendix B shows the systematic method by which the attractor equations for the FHSV model is solved. This procedure can be used for any model which has the following vector multiplet moduli space

$$\mathcal{M}_V = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times \mathrm{SO}(n)}. \tag{1.3}$$

2. Two $\mathcal{N} = 2$ theories

In this section we review the construction of the two $\mathcal{N} = 2$ models that we will be studying. The dyons which will be the focus of our interest preserve 1/2 of the 8 supersymmetries of these theories.

2.1 The STU model

This model is best described in terms of a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of type IIB [25, 26] (example C in [25]). Consider Type IIB compactified on $T^4 \times S^1 \times \tilde{S}^1$, the first $\mathbb{Z}_2$, g_1 : acts by $(-1)^{F_L}$ together with a half shift on S^1, the second $\mathbb{Z}_2$, g_2 acts as an inversion of the coordinates on T^4 together with a half shift on $\tilde{S}^1$. The theory with only with the g_1 action is the same as the one considered in [10], this is a $\mathcal{N} = 4$ theory. The second $\mathbb{Z}_2$ action g_2 further breaks the supersymmetry down to $\mathcal{N} = 2$. This theory has 3 vector multiplets and 4 hyper multiplets. The T-duality symmetry of this theory is $\mathrm{SO}(2,2;\mathbb{Z})$. In the type IIA description of this orbifold the dilaton belongs to the hypermultiplet, while in the type

IIB description[2] it belongs to the vector multiplet [25, 26]. Therefore the vector multiplet moduli space is not corrected by quantum corrections and it is given by

$$\mathcal{M}_V = \left.\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right|_S \times \left.\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right|_T \times \left.\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right|_U, \tag{2.1}$$

where S refers to the axion dilaton moduli and T and U refer to the Kähler and complex structure of the torus $S \times \tilde{S}$. This theory is invariant under the symmetry $\Gamma_S(2) \times \Gamma_T(2) \times \Gamma_{\mathrm{U}}(2)$ where $\Gamma(2)$ is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1, \quad b, c \in 2\mathbb{Z}, \; a, d \in 2\mathbb{Z}+1. \tag{2.2}$$

It also has the triality invariance $S \leftrightarrow T \leftrightarrow -1/U$.

The dyons we consider are the twisted sector dyons of the parent $\mathcal{N} = 4$ theory which are preserved by the second orbifold projection g_2. Thus these dyons have electric, and magnetic charges only along S^1 and $\tilde{S}^1$ directions. Since these dyons are 1/4 BPS states in the parent theory they will preserve 1/2 of the supersymmetries of the daughter theory. For the purposes of obtaining the subleading corrections to the entropy of the dyons we will need the coefficient of the Gauss-Bonnet term. The Gauss-Bonnet term is given by following combinations of 4 derivative terms made of the curvature tensor

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{2.3}$$

The coefficient of this term for the STU model was evaluated in [26], this is given by

$$\begin{aligned} \frac{1}{128\pi^2}\Bigg(4\ln\left(\frac{M_p^2}{p^2}\right) &- 4\log\left[\vartheta_2(\tau)^2\vartheta_2(-\bar{\tau})^2(\tau_2)\right] \\ &-4\log\left[\vartheta_2(y^+)^2\vartheta_2(-\bar{y}^+)^2(y_2^+)\right] - 4\log\left[\vartheta_4(y^-)^2\vartheta_4(-\bar{y}^-)^2(y_2^-)\right]\Bigg), \end{aligned} \tag{2.4}$$

where τ refers to the complex combination of the axion and dilaton moduli given by

$$\tau = -a + iS, \tag{2.5}$$

where a is the axion and $S = \exp(-2\phi)$ is the dilaton. y^+ and y^- refer to the Kähler and complex structure of the $S \times \tilde{S}$ torus, which we have denoted as the T and U moduli.[3] The normalization of the Gauss-Bonnet term is determined as follows [29]: Let the axion-dilaton dependence of the Gauss-Bonnet term be given by

$$-\frac{\mathcal{K}}{128\pi^2}\ln\left[\tau_2 f(\tau) f(-\bar{\tau})\right]. \tag{2.6}$$

[2]Here the duality symmetry which relates the type IIA and the type IIB description of the theory is not the conventional T-duality but the one which is part of the U-duality group and is a strong-weak duality symmetry [25]

[3]Throughout the paper, the subscripts 1 and 2 on complex moduli refer to its real and imaginary parts respectively.

The functional dependence of the term in the square bracket is invariant under the S-duality symmetry $\Gamma(2)_S$. Then the coefficient $\mathcal{K}$ is given by the number of harmonic p forms of T^4 left invariant under the action of g_1 and g_2 weighted by $(-1)^p$. g_2 projects retains only the even forms which results in 8 forms. Since g_1 reduces the supersymmetry by projecting out the left moving fermions, it project out the 3 self-dual 2-forms and a combination of the zero form and 4 form. Thus out of the 8 forms 4 are left invariant. We therefore conclude $\mathcal{K} = 4$. Once this normalization is determined the dependence of the axion-dilaton moduli and the other moduli is obtained from the calculation of [26], equation (5.3). The dependence of the coefficient of the Gauss-Bonnet term on $\ln(M_p^2/p^2)$ where M_p is the Planck's constant and p^2 is the graviton momentum is due to the trace anomaly of the theory. Though the coefficient of the trace anomaly does not play a role in the evaluation of the sub-leading contribution to the entropy, its contribution to the coefficient of the Gauss-Bonnet term for a general $\mathcal{N} = 2$ theory is given by

$$-\frac{1}{128\pi^2}\left(\frac{23 + n_h - n_v}{6}\right)\ln\left(\frac{M_p^2}{\Lambda^2}\right), \tag{2.7}$$

where n_h is the number of hypermultiplets and n_v is the number of vector multiplets, for the STU model $n_h - n_v = 1$. Note that the coefficient of the Gauss-Bonnet term (2.4) is invariant under the triality symmetry $\tau \leftrightarrow y^+ \leftrightarrow -1/y^-$ as well as $\Gamma_S(2) \times \Gamma_T(2) \times \Gamma_{\rm U}(2)$, which is the symmetry of the model. This completes the explantation of all the terms in (2.4).

2.2 The FHSV model

For our purposes it is easiest to describe the FHSV model as a freely acting orbifold of the heterotic $E_8 \times E_8$ theory. Consider the heterotic string on the following even, self-dual Lorentzian lattice $\Gamma^{(22,6)}$ of the form

$$\Gamma^{(9,1)} \oplus \Gamma^{(9,1)} \oplus \Gamma^{(1,1)} \oplus \Gamma^{(1,1)} \oplus \Gamma^{(2,2)}. \tag{2.8}$$

Here the two $\Gamma^{(9,1)}$ factors are isomorphic. We now orbifold by a $\mathbb{Z}_2$ action which exchanges the first two factors and act as -1 on the third $\Gamma^{(1,1)}$ and the last $\Gamma^{(2,2)}$ together with the half shift in the fourth $\Gamma^{(1,1)}$ factor. The T-duality group of this heterotic string is $\mathrm{SO}(2, 10; \mathbb{Z})$, the moduli space of this theory is known exactly [27]. For purposes of this paper it is sufficient to focus on the vector multiplet moduli space. There are 11 vector multiplets in this theory and it moduli space is given by

$$\mathcal{M}_V = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,10)}{\mathrm{SO}(2) \times \mathrm{SO}(10)}. \tag{2.9}$$

Note that the coset $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ which is parameterized by the axion-dilaton moduli factorizes from the rest of the vector multiplet moduli, we call these moduli the T-moduli. The S duality of this theory is $\Gamma_S(2)$.

We focus on a class of dyon configurations which preserves 1/2 of the remaining symmetry in the FHSV model. These class of dyons are obtained by embedding the 1/4 BPS dyon of the parent $\mathcal{N} = 4$ theory such that the dyon configuration is preserved by the FHSV orbifold action. We now describe the charge configuration of such a dyon: Let us call the unprojected combination of the two $\Gamma^{(9,1)}$ lattices as $\Gamma_I^{(9,1)}$, we denote the fourth $\Gamma^{(1,1)}$ as $\Gamma_S^{(1,1)}$. We focus on dyons which have electric and magnetic charges values on the lattices $\Gamma_I^{(9,1)} \oplus \Gamma_S^{(1,1)}$ which have twisted sector charges along the $\Gamma_S^{(1,1)}$ lattice. It is clear that such charge configurations are preserved by the FHSV orbifold action, since these configurations do not have charges in the lattices $\Gamma^{(1,1)} \oplus \Gamma^{(2,2)}$ on which the FHSV orbifold acts as an inversion.

We will study the macroscopic entropy of these dyons using the entropy function formalism to the first subleading term. For this purpose we will need the coefficient of the Gauss-Bonnet term in this model. The dilaton-axion and the T-moduli dependence of the Gauss-Bonnet coefficient of this model was evaluated exactly in [28]. We now review their result: The coefficient of this term for $\mathcal{N} = 2$ compactifications is given by

$$\frac{1}{128\pi^2}\left(4\ln\left(\frac{M_p^2}{p^2}\right) + \mathcal{F}_1(\tau, \bar{\tau}, y, \bar{y})\right). \tag{2.10}$$

In (2.10) $\mathcal{F}_1(\tau, \bar{\tau})$ is the modular function determined by the threshold calculation and τ is the complex combination of the axion and the dilaton given in (2.5). For the FHSV model, the τ and the T-moduli dependence of the threshold corrections is given by [28]

$$\begin{aligned}\mathcal{F}_1(\tau, \bar{\tau}, y, \bar{y}) = & -12\log\left[\eta(2\tau))(\eta(-2\bar{\tau})(\tau_2)\right] \\ & -\log\left[\Phi_{\rm BE}(y)\Phi_{\rm BE}(-\bar{y})(4y_2^+y_2^- - \vec{y}_2^2)^4\right],\end{aligned} \tag{2.11}$$

where $\Phi_{\rm BE}(y)$ is the Borcherds-Enriques modular form of weight four on the discrete group $E_8 \times \Gamma^{1,1}(-2) \times \Gamma^{1,1}(-2)$. y refer to the 10 complex T-moduli, $y = \{y^+, y^-, y^i\}, i = 1, \cdots 8$. The subscript 2 on the various moduli in (2.11) stands for the imaginary values the moduli. For the purposes of this paper will not need the details of the Borcherds-Enriques form, please see [28] for the details. Just as in the STU model we fix the normalization of the coefficient of the Gauss-Bonnet term by examining the axion-dilaton dependence. This is given by (2.6), here $\mathcal{K}$ is the number of p-forms invariant on the dual description of the the FHSV orbifold in type IIA theory on $K3$ weighted by $(-1)^p$. This turns out to be 12, the FHSV orbifold action projects out 12 of the 24 forms of $K3$. The dependence of the axion dilaton and the remaining T-moduli dependence is then read out from [28]. The trace anomaly dependence is given by (2.7) with $n_v = 11$ and $n_h = 12$ for the FHSV model.

3. Entropy function for dyons

In this section we will evaluate the first sub-leading contribution to the Hawking-Bekenstein entropy due to Gauss-Bonnet term in the effective action for dyons in both the STU model and the FHSV model discussed in the previous section. We will follow the entropy function approach developed in [30]. In this approach to find the first sub-leading contribution to the entropy we will need both the coefficient of the Gauss-Bonnet term as well as the attractor values of vector multiplet moduli at the two derivative level. We first evaluate the attractor values of all the vector multiplet moduli to both these models and then substitute these values in the coefficient of the respective Gauss-Bonnet term in these models to evaluate the sub-leading contribution to the Hawking-Bekenstein entropy.

To make the discussion self contained we briefly review the entropy function formalism. Consider an extremal black hole solution in any of the $\mathcal{N}=2$ theories of interest in this paper. The near horizon geometry of this black holes is given by

$$\begin{aligned} ds^2 &= v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2(d\theta^2+\sin^2\theta d\phi^2), \\ S &= u_S, \qquad a=u_a, \qquad \tilde{M}_{ij}=u_{Mij}, \\ F^{(i)}_{rt} &= e_i, \qquad F^{(i)}_{\theta\phi}=\frac{p_i}{4\pi}. \end{aligned} \tag{3.1}$$

Here $i=1,\cdots n+2$ and the near horizon vector multiplet moduli, $n=10$ for the FHSV model and $n=2$ for the STU model. u_{Mij} satisfies

$$u_M^T L u_M = L, \quad u_M^T = u_M, \tag{3.2}$$

where L is the Lorentzian metric with n negative signatures and 2 positive signatures; $L=\mathrm{Dia}(-1,-1,\cdots-1,+1,+1)$. Substituting the solution (3.1) in the supergravity action we obtain [30]

$$\begin{aligned} &f(u_s,u_a,u_M,\vec{v},\vec{e},\vec{p}) \equiv \int d\theta d\phi\sqrt{-\mathrm{det}G}\mathcal{L} \\ &= \frac{1}{8}v_1v_2u_S\left[-\frac{2}{v_1}+\frac{2}{v_2}+\frac{2}{v_1^2}e_i(Lu_ML)_{ij}e_j-\frac{1}{8\pi^2v_2^2}p_i(Lu_ML)_{ij}p_j+\frac{u_a}{\pi u_S v_1v_2}e_iL_{ij}p_j\right]. \end{aligned} \tag{3.3}$$

We then obtain the electric charges q_i given by

$$q_i \equiv \frac{\partial f}{\partial e_i} = \frac{v_2u_S}{2v_1}(Lu_ML)_{ij}e_j+\frac{u_a}{8\pi}L_{ij}p_j. \tag{3.4}$$

Evaluating the Legendre transform with respect to the variables e_i we obtain the entropy function F

$$\begin{aligned} F(u_S,u_a,u_M,\vec{v},\vec{q},\vec{p}) &\equiv 2\pi(e_ip_i - f(u_S,u_a,u_M,\vec{v},\vec{e},\vec{p})) \\ &= 2\pi\left[\frac{u_S}{4}(v_2-v_1)+\frac{v_1}{v_2u_S}q^Tu_Mq+\frac{v_1}{64\pi^2v_2u_s}(u_S^2+u_a^2)Lu_MLp\right. \\ &\qquad \left.-\frac{v_1}{4\pi v_2u_S}u_aq^Tu_MLp\right]. \end{aligned} \tag{3.5}$$

We define the charge vectors

$$Q_i = 2q_i, \qquad P_i=\frac{1}{4\pi}L_{ij}p_j, \tag{3.6}$$

so that P_i and Q_i are integers. Substituting these definitions of charges in the entropy function we obtain

$$F=\frac{\pi}{2}\left[u_S(v_2-v_1)+\frac{v_1}{v_2u_S}\left(Q^Tu_MQ+(u_S^2+u_a^2)P^Tu_MP-2u_aQ^Tu_MP\right)\right]. \tag{3.7}$$

Now to evaluate the Hawking-Bekenstein entropy we need to find the the extremum of the above function with respect to the moduli $u_S, u_a, u_{Mij}, v_1, v_2$. For this we can first minimize the entropy function in (3.7) with respect to v_1 and v_2. It is easy to see from the dependence of the entropy function on v_1 and v_2, the minimum occurs at $v_1 = v_2$. We then organize the axion and the dilaton moduli in terms of the complex scalar τ defined as

$$\tau = -u_a + iu_S. \tag{3.8}$$

Using τ and the parameterization and $v_1 = v_2$, the entropy function (3.7) reduces to

$$F = \frac{\pi}{2\tau_2}\left[(Q + \tau P)^T u_M (Q + \tau P)\right]. \tag{3.9}$$

In the next two subsections we use the above entropy function and to further minimize with respect to the T-moduli parameterized by u_M and the axion dilaton moduli τ for the STU model and the FHSV model. In both the cases we obtain the attractor values of these moduli in terms of the charges explictly. We then use the attractor values of the moduli to evaluate the subleading correction due to the Gauss-Bonnet term.

3.1 The two derivative entropy function: STU model

For the case of the STU model the moduli matrix u_M is a 4×4 matrix which parameterizes the coset $\mathrm{SO}(2,2)/(\mathrm{SO}(2)\times\mathrm{SO}(2))$. It satisfies the conditions

$$u_M^T = u_M, \qquad u_M^T L u_M = u_M, \tag{3.10}$$

where L is the diagonal metric $L = \mathrm{Dia}(1,1,-1,-1)$. From the conditions in (3.10) it can be easily seen that there are 4 independent variables which parameterize the matrix u_M. A convenient parameterization is as follows: We first introduce 4 complex numbers satisfying

$$w_1^2 + w_2^2 - w_3^2 - w_4^2 = 0, \tag{3.11}$$

together with the identification $w_I \sim cw_I$ where c is a complex number. Thus w_I's parameterize the coset $\mathrm{SO}(2,2)/(\mathrm{SO}(2)\times\mathrm{SO}(2))$. Using the scaling symmetry we can solve the constraint (3.11) by introducing complex numbers y^+ and y^- and writing w_I's as

$$\begin{aligned} w_1 &= -1 + y^+y^-, \qquad & w_2 &= y^+ + y^-, \\ w_3 &= y^+ - y^-, \qquad & w_4 &= 1 + y^+y^-. \end{aligned} \tag{3.12}$$

Note that we have use the scaling symmetry of w_I's to set $w_4 - w_1 = 2$. Using the above solution of the constraint (3.11) it can be seen

$$|w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = 2Y, \tag{3.13}$$

where $Y = 4y_2^+y_2^-$ is related to the Kähler potential on the moduli space by

$$K = -\log Y. \tag{3.14}$$

Now we can parameterize the moduli matrix u_M by

$$u_M = L\tilde{U}L - L, \tag{3.15}$$

where $\tilde{U}$ is given by

$$\tilde{U} = \frac{w_I \bar{w}_J + \bar{w}_I w_J}{Y}. \tag{3.16}$$

It can be seen that using (3.12), (3.13) and the definitions (3.15) and (3.16) the conditions on u_M given in (3.10) are satisfied. Note that now we have parameterized the matrix u_M in terms of y^+ and y^-. Substituting this parameterization in the entropy function (3.9) we obtain

$$F = \frac{\pi}{2}\left[\frac{|(Q+\tau P)\cdot w|^2}{\tau_2 Y} + \frac{|(Q+\bar{\tau}P)\cdot w|^2}{\tau_2 Y} - \frac{(Q+\tau P)\cdot(Q+\bar{\tau}P)}{\tau_2}\right], \tag{3.17}$$

where dot product $\cdot$ is with respect to the metric L. To determine the values of the T and U-moduli (y^+, y^- respectively) at the attractor point it is sufficient to focus terms on the first two terms in (3.17). The first two terms are identical except for the exchange of $\tau \leftrightarrow \bar{\tau}$ in the numerator. Our strategy for minimizing with respect to y^+ and y^- is as follows: We will first just focus on the first term

$$F_T = \frac{\pi}{2}\left[\frac{|(Q+\tau P)\cdot w|^2}{\tau_2 Y}\right], \tag{3.18}$$

and minimize this term with respect to y^+ and y^-. We will see that the attractor values of the moduli y^+ and y^- are independent of the axion-dilaton moduli τ. Therefore these values of the attractor moduli minimize the second term in (3.17) simultaneously, since the second term is same as the first term with $\tau \to \bar{\tau}$. Thus to minimize with respect to the y^+ and y^- moduli it is sufficient to focus on (3.18).

To write out (3.18) in terms of the moduli y^+ and y^- it is convenient to define the following variables

$$\begin{array}{llll} Q^1+Q^4 = N_1, & Q^1-Q^4 = W_1, & Q^2+Q^3 = N_2, & Q^2-Q^3 = W_2, \\ P^1+P^4 = \tilde{N}_1, & P^1-P^4 = \tilde{W}_1, & P^2+P^3 = \tilde{N}_2, & P^2-P^3 = \tilde{W}_2. \end{array} \tag{3.19}$$

We also define the various components of the complex combination $Q+\tau P$ as follows

$$\begin{array}{ll} Q^1+Q^4+\tau(P^1+P^4) = N_1+\tau\tilde{N}_1 = n_1, & Q^1-Q^4+\tau(P^1-P^4) = W_1+\tau\tilde{W}_1 = w_1', \\ Q^2+Q^3+\tau(P_2+P_3) = N_2+\tau\tilde{N}_2 = n_2, & Q^2-Q^3+\tau(P_2-P_3) = W_2+\tau\tilde{W}_2 = w_2'. \end{array} \tag{3.20}$$

Using the variables the T-moduli dependent part of the entropy function in (3.17) is given by

$$F_T = \frac{\pi}{8\tau_2 y_2^+ y_2^-}\left|-n_1 + y^+y^-w_1' + y^+w_2' + y^-n_2\right|^2, \tag{3.21}$$

where the subscript T, U indicates the term dependent on the T, U moduli of the entropy function in (3.17). Minimizing with respect to y^+ and y^- we obtain the following equations respectively

$$y^- \bar{y}^+ w_1' + w_2' \bar{y}^+ + y^- n_2 - n_1 = 0, \qquad (3.22)$$
$$y^+ \bar{y}^- w_1' + n_2 \bar{y}^- + y^+ w_2' - n_1. = 0$$

Eliminating y^- from the above equations we obtain the following quadratic equation for y^+

$$(\tilde{W}_2 W_1 - W_2 \tilde{W}_1)(y^+)^2 + [(\tilde{W}_2 N_2 - W_2 \tilde{N}_2) - (\tilde{N}_1 W_1 - N_1 \tilde{W}_1)] y^+ - (\tilde{N}_1 N_2 - N_1 \tilde{N}_2) = 0. \quad (3.23)$$

Note that in this quadratic equation which determines the moduli y^+, the axion dilaton dependence completely drops out. To obtain the solution for y^+ we can simplify the discriminant

$$D = \Big[(\tilde{W}_2 N_2 - W_2 \tilde{N}_2) - (\tilde{N}_1 W_1 - N_1 \tilde{W}_1)\Big]^2 + 4(\tilde{W}_2 W_1 - W_2 \tilde{W}_1)(\tilde{N}_1 N_2 - N_1 \tilde{N}_2),$$
$$= 4((Q\cdot P)^2 - Q^2 P^2). \qquad (3.24)$$

Here we have re-written the charges in terms of Q's and P's using the relations in (3.19). Since we are looking at supersymmetric black holes we have $Q^2P^2 - (Q\cdot P)^2 > 0, Q^2 > 0, P^2 > 0$, this implies the discriminant D is always negative. Thus the solution for y^+ is always complex, the real and imaginary parts of y^+ are given by

$$y_1^+ = -\frac{(\tilde{W}_2 N_2 - W_2 \tilde{N}_2) - (\tilde{N}_1 W_1 - N_1 \tilde{W}_1)}{2(\tilde{W}_2 W_1 - W_2 \tilde{W}_1)}, \qquad (3.25)$$
$$y_2^+ = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{(\tilde{W}_2 W_1 - W_2 \tilde{W}_1)}.$$

Note that finally we have to choose the solution with $y_2^+ > 0$, we choose $\tilde{W}_2 W_1 - W_2 \tilde{W}_1 > 0$. For later purpose it is convenient to write the above solution more suggestively as follows. The electric and magnetic charges are 4-vectors in $\mathrm{SO}(2,2;\mathbb{Z})$. Let us consider the antisymmetric combination of these vectors

$$T^{ij} = Q^i P^j - Q^j P^i, \qquad (3.26)$$

where $i,j = 1,\cdots 4$, these 6 components are S-duality invariants. We can project them into components which transform as a **3** of the left $\mathrm{SO}(2,1;\mathbb{Z})$ of $\mathrm{SO}(2,2;\mathbb{Z})$ and **3** of the right $\mathrm{SO}(2,1;\mathbb{Z})$ using the self dual and anti-self dual 't Hooft symbols of $\mathrm{SO}(2,2)$ respectively. From the definition of the 't Hooft symbols in (A.1) and (A.2) we see that the solution (3.25) can be written as

$$y_1^+ = -\frac{\eta_{1;ij}T^{ij}}{\eta_{+i;j}T^{ij}}, \quad y_2^+ = \frac{\sqrt{\eta_{a;ij}\eta^a_{kl}T^{ij}T^{kl}}}{\eta_{+;ij}T^{ij}}. \qquad (3.27)$$

where $a = 1,2,3$. Note that we have used the identity (A.6) to rewrite the invariant $Q^2P^2 - (Q\cdot P)^2$ in the above equation. A similar analysis for the y^- yields the quadratic equation

$$(\tilde{N}_2 W_1 - N_2 \tilde{W}_1)(y^-)^2 + \Big[(\tilde{N}_2 W_2 - N_2 \tilde{W}_2) + (\tilde{W}_1 N_1 - \tilde{N}_1 W_1)\Big] y^- + (\tilde{W}_2 N_1 - W_2 \tilde{N}_1) = 0, \qquad (3.28)$$

here again the axion dilaton dependence drops out. The discriminant of this quadratic equation is given by

$$\begin{aligned} D &= \Big[(\tilde{N}_2 W_2 - N_2 \tilde{W}_2) + (\tilde{W}_1 N_1 - \tilde{N}_1 W_1)\Big]^2 - 4(\tilde{N}_2 W_1 - N_2 \tilde{W}_1)(\tilde{W}_2 N_1 - W_2 \tilde{N}_1), \\ &= 4(Q\cdot P)^2 - Q^2 P^2. \end{aligned} \tag{3.29}$$

Thus the real and the imaginary parts of y^- are given by

$$\begin{aligned} y_1^- &= -\frac{(\tilde{N}_2 W_2 - N_2 \tilde{W}_2) + (\tilde{W}_1 N_1 - \tilde{N}_1 W_1)}{2(\tilde{N}_2 W_1 - N_2 \tilde{W}_1)}, \\ y_2^- &= \frac{\sqrt{Q^2 P^2 - (Q\cdot P)^2}}{\tilde{N}_2 W_1 - N_2 \tilde{W}_1}. \end{aligned} \tag{3.30}$$

Again we have to choose the solution with $y_2^- > 0$ for that we have $\tilde{N}_2 W_1 - N_2 \tilde{W}_1 > 0$. Rewriting the solution (3.30) in terms of the 't Hooft symbols we obtain

$$y_1^- = -\frac{\tilde{\eta}_{1;ij} T^{ij}}{\tilde{\eta}_{-;ij} T^{ij}}, \qquad y_2^- = \frac{\sqrt{\tilde{\eta}_{a;ij} \tilde{\eta}^a_{kl} T^{ij} T^{kl}}}{\tilde{\eta}_{-;ij} T^{ij}}. \tag{3.31}$$

For later convenience we also write down the real and imaginary parts of $-1/y^-$

$$\begin{aligned} -\left(\frac{1}{y^-}\right)_1 &= \frac{(\tilde{N}_2 W_2 - N_2 \tilde{W}_2) + (\tilde{W}_1 N_1 - \tilde{N}_1 W_1)}{2(\tilde{W}_2 N_1 - W_2 \tilde{N}_1)} = \frac{\tilde{\eta}_{1;ij} T^{ij}}{\tilde{\eta}_{+;ij} T^{ij}}, \\ -\left(\frac{1}{y^-}\right)_2 &= \frac{\sqrt{Q^2 P^2 - (Q\cdot P)^2}}{\tilde{W}_2 N_1 - W_2 \tilde{N}_1} = \frac{\sqrt{\tilde{\eta}_{a;ij} \tilde{\eta}^a_{kl} T^{ij} T^{kl}}}{\tilde{\eta}_{+;ij} T^{ij}}. \end{aligned} \tag{3.32}$$

The solutions given in (3.25) and (3.30) are independent of the axion-dilaton moduli and thus minimize both the terms which depend on y^+ and y^- in (3.17) and therefore are the attractor values for the entropy function. The solution for the attractor values of the moduli has been derived earlier for special charge configurations [30, 31]. The analysis given above is the complete solution for arbitrary charge configurations, it reduces to the solutions found in [30, 31] for the respective charge configurations. The general analysis enabled us to write down the attractor values of the moduli in a manifestly symmetric form given in (3.27), (3.31).

Substituting the attractor values of the T and U moduli given in (3.27) and (3.31) in the entropy function (3.17) we obtain

$$F|_{T,U,\min} = \frac{\pi}{2\tau_2}\left[(Q+\tau P)\cdot(Q+\bar{\tau}P)\right]. \tag{3.33}$$

We now can further minimize with respect to the axion dilaton moduli to obtain

$$\tau_1 = -\frac{Q\cdot P}{P^2}, \qquad \tau_2 = \frac{\sqrt{Q^2 P^2 - (Q\cdot P)^2}}{P^2}. \tag{3.34}$$

Finally substituting the values of all the vector multiplet moduli in the entropy function we get the usual Hawking-Bekenstein entropy

$$F|_{\min} = \pi\sqrt{Q^2 P^2 - (Q\cdot P)^2}. \tag{3.35}$$

Note that all the vector multiplets y^+, y^-, τ are fixed at the attractor values, therefore this attractor point has no flat directions.

3.2 The two derivative entropy function: FHSV model

Let us call all the vector multiplet moduli other than the axion-dilaton moduli as T-moduli. The T-moduli in the FHSV model parameterizes the coset

$$\mathcal{M}_T = \frac{\mathrm{SO}(10,2)}{\mathrm{SO}(10)\times \mathrm{SO}(2)}, \tag{3.36}$$

To explictly find the values of the T-moduli at the attractor point we first need to parameterize the 12×12 moduli matrix u_M which satisfies the condition

$$u_M^T = u_M, \qquad u_M^T L u_M = L. \tag{3.37}$$

where $L = \mathrm{Dia}(-1, \cdots, -1, 1, 1)$ is the Lorentzian metric with 10 negative 1's and 2 positive 1's . From the conditions in (3.37) the number of independent variables required to parameterize u_M is 20. Just as in the STU model we first introduce 10+2 complex numbers satisfying

$$-\sum_{I=1}^{10} w_I^2 + w_{11}^2 + w_{12}^2 = 0, \tag{3.38}$$

together with the identification $w_I \sim c w_I$, where c is a complex number. Note that the constraint in (3.38) and the identification of w's upto complex scalings reduce the number of independent parameters to 20 which is the required number of variables to parameterize the moduli space in (3.36). Using the scaling degree of freedom the constraints in (3.38) can be solved by introducing the 10 complex numbers $(y^+, y^-, \vec{y})$ where $\vec{y}$ is a 8 dimensional vector. These variables are related to the 12 w_I's by

$$\begin{aligned}
w_I &= y_I, \quad I = 1\cdots 8, & w_9 &= \frac{1}{\sqrt{2}}(y^+ - y^-), \\
w_{10} &= 1 + \frac{y^2}{4}, & w_{11} &= \frac{1}{\sqrt{2}}(y^+ + y^-), \\
w_{12} &= -1 + \frac{y^2}{4}, & y^2 &= 2y^+y^- + \vec{y}^2.
\end{aligned} \tag{3.39}$$

On substituting these values of w_I in (3.38) it is easy to see that the constraint is satisfied. The above parameterization amounts to scaling $w_{10} - w_{12}$ such that its value is constant given by $w_{10} - w_{12} = 2$. Using the above solution of the constraint (3.38) it can be seen that

$$\begin{aligned}
-\sum_{I=1}^{10} |w_I|^2 + |w_{11}|^2 + |w_{12}|^2 &= 2Y, \\
\text{where } Y = (\mathrm{Im} y)^2 &= 2y_2^+ y_2^- - \vec{y}_2^2.
\end{aligned} \tag{3.40}$$

Y is related to the Kähler potential on the moduli space by

$$K = -\log Y. \tag{3.41}$$

The variables y to parameterize the vector multiplet moduli was introduced in [28, 32]. We now parameterize the vector multiplet moduli matrix as

$$u_M = L\tilde{U}L - L. \tag{3.42}$$

The conditions on the moduli matrix given in (3.2) result in the following conditions on $\tilde{U}$

$$\tilde{U}^T = \tilde{U}, \qquad \tilde{U}L\tilde{U} - 2\tilde{U} = 0. \tag{3.43}$$

We can now further parameterize $\tilde{U}$ as

$$\tilde{U} = \frac{w_I \bar{w}_J + \bar{w}_I w_J}{Y}, \tag{3.44}$$

here $I = 1 \cdots 12$ It is easy to see that given the constraints (3.38) on the w's we see that the conditions on $\tilde{U}$ in (3.43) are satisfied.

Using τ to parameterize the axion dilaton moduli as in (3.8) and parameterizing the moduli matrix u_M in terms of w from (3.42), (3.44) in the entropy function (3.9) we can write the entropy function as

$$\begin{aligned} F &= \frac{\pi}{2\tau_2}\left[(Q+\tau P)u_M(Q+\bar{\tau}P)\right], \\ &= \frac{\pi}{2}\left[\frac{|(Q+\tau P)\cdot w|^2}{\tau_2 Y} + \frac{|(Q+\bar{\tau}P)\cdot w|^2}{\tau_2 Y} - \frac{(Q+\tau P)\cdot(Q+\bar{\tau}P)}{\tau_2}\right]. \end{aligned} \tag{3.45}$$

Note that in the above the dot product is taken with the metric L, From here it is easy to see that F is both SL$(2,R)$ as well as SO$(10,2;R)$ invariant. We first minimize the entropy function with respect to the T-moduli $(y^+, y^-, \vec{y})$ for right moving charges. Right moving charges are defined by the condition $Q\cdot Q > 0, P\cdot P > 0$. Generic supersymmetric dyons are right moving. Details of the minimization procedure are provided in the appendix. To write down the solution we define the following combination of charges

$$\begin{aligned} &Q^{12}+Q^{10} = N_1, \quad Q^{12}-Q^{10} = W_1, \quad Q^{11}+Q^9 = N_2, \quad Q^{11}-Q^9 = W_2, \\ &P^{12}+P^{10} = \tilde{N}_1, \quad P^{12}-P^{10} = \tilde{W}_1, \quad P^{11}+P^9 = \tilde{N}_2, \quad P^{11}-P^9 = \tilde{W}_2. \end{aligned} \tag{3.46}$$

We now write down the solution of all the T-moduli: y^i, $i = 1, \cdots 8$ and y^- are determined in terms of y^+ from the equation

$$\begin{aligned} y^i &= \sqrt{2}y^+ \frac{Q^i\tilde{W}_1 - P^i W_1}{N_2\tilde{W}_1 - \tilde{N}_2 W_1} + 2\frac{\tilde{N}_1 Q^i - N_2 P^i}{N_2\tilde{W}_1 - \tilde{N}_2 W_1}, \\ y^- &= y^+ \frac{W_2\tilde{W}_1 - \tilde{W}_2 W_1}{N_2\tilde{W}_1 - \tilde{N}_2 W_1} + \sqrt{2}\frac{\tilde{N}_2 W_2 - N_2\tilde{W}_2}{N_2\tilde{W}_1 - \tilde{N}_2 W_1}, \end{aligned} \tag{3.47}$$

y^+ is determined from the solution of the following quadratic equation

$$A\left(\frac{y^+}{\sqrt{2}}\right)^2 + B\frac{y^+}{\sqrt{2}} + C = 0, \tag{3.48}$$
$$A = \tilde{W}_2 W_1 - W_2\tilde{W}_1 + \sum_{i=1}^{8}\frac{(Q^i\tilde{W}_1 - P^i W_1)^2}{N_2\tilde{W}_1 - \tilde{N}_2 W_1},$$
$$B = (\tilde{W}_2 N_2 - W_2\tilde{N}_2) - (\tilde{N}_1 W_1 - N_1\tilde{W}_1) + \sum_{i=1}^{8}\frac{(\tilde{N}_2 Q^i - P^i N_2)(\tilde{W}_1 Q^i - P^i W_1)}{N_2\tilde{W}_1 - \tilde{N}_2 W_1},$$
$$C = N_1\tilde{N}_2 - \tilde{N}_1 N_2.$$

Note that the coefficients which determine the attractor values of the moduli y^+ and y^i in (3.47) and (3.48) are all functions of the S-duality invariants $Q^I P^J - P^I Q^J$, with $I, J = 1, \cdots 12$. Therefore the attractor values of the T-moduli do not transform under the S-duality symmetry of the FHSV model. As a small check on the above solutions for the T-moduli, note that on setting the charges $Q^i = 0, P = 0$ the equations (3.48) reduce to (3.23) with the $y^+ \to \sqrt{y^+}$. Furthermore from the solutions it is easy to see that under the scaling $Q^I \to \lambda Q^I,\ P^I \to \tilde{\lambda} P^I$ the attractor values of the T-moduli remain invariant. Thus for independent scalings of the electric and the magnetic charges the attractor values of the T-moduli are invariant. From (3.48), it is see that the attractor values of y^+ are functions of ratios of $B/A, C/A$, thus they are of $\mathcal{O}(Q^0, P^0)$ even if the charges electric and magnetic charges are scaled with $\lambda \gg 1, \tilde{\lambda} \gg 1$. We now substitute the values of the T-moduli into the entropy function given in (3.45), this gives

$$F = \frac{\pi}{2\tau_2}\left[(Q + \tau P)\cdot(Q + \bar{\tau} P)\right]. \tag{3.49}$$

Now we proceed with minimizing the above function with respect to the axion-dilaton moduli, this results in the following attractor values

$$\tau_1 = -\frac{Q\cdot P}{P^2}, \qquad \tau_2 = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}. \tag{3.50}$$

Finally substituting all the attractor values of the vector multiplet moduli into the entropy function we obtain the following Hawking-Bekenstein entropy

$$F = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2}. \tag{3.51}$$

3.3 Including the Gauss-Bonnet correction

In this subsection we include the contribution of the Gauss-Bonnet term to the entropy function and evaluate its contribution to the black hole entropy. We will retain terms to $\mathcal{O}(Q^0, P^0)$ terms in charges and neglect contributions at $\mathcal{O}(1/Q^2, 1/P^2)$. The coefficient of the Gauss-Bonnet in both the STU model (2.4) and the FHSV model (2.10) contains a term proportional to the trace anomaly which depends on graviton momentum p^2. Since this term does not depend on the moduli of the theory the contribution to the Gauss-Bonnet term can be neglected. Retaining this term in the entropy function formalism just shifts the

Entropy by a charge independent constant which we do not keep track of. Therefore we can restrict our attention to the moduli dependent terms in the coefficient of the Gauss-Bonnet term in (2.4) and (2.10). Consider first a coefficient of the Gauss-Bonnet term to be

$$\frac{1}{128\pi^2}\mathcal{F}(\mu,\bar{\mu}), \tag{3.52}$$

where μ refers to the axion-dilaton moduli together with the rest of the vector multiplet moduli. The change in the entropy function due to the above coefficient is (3.52) is

$$\Delta F = \frac{1}{2}\mathcal{F}(\mu,\bar{\mu}). \tag{3.53}$$

The total entropy function is now given by

$$F = \lambda^2 F^{(2)}(\mu,\bar{\mu}) + \frac{1}{2}\mathcal{F}(\mu,\bar{\mu}), \tag{3.54}$$

where $F^{(2)}$ refers to the two derivative entropy function discussed in the previous section. The two derivative term is proportional to $\mathcal{O}(Q^2,P^2)$, this is indicated by the coefficient λ^2 in (3.54). We can now minimize with respect to the moduli μ and solve for the moduli in powers of $\frac{1}{\lambda^2}$, we denote this correction as

$$\mu = \mu^* + \frac{1}{\lambda^2}\delta\mu + \mathcal{O}(1/\lambda^4), \tag{3.55}$$

where μ^* solves the attractor equations of the two derivative entropy function $F^{(2)}$ given by

$$\left.\frac{\partial F^{(2)}(\mu,\bar{\mu})}{\partial\mu}\right|_{\mu=\mu^*} = 0, \qquad \left.\frac{\partial F^{(2)}(\mu,\bar{\mu})}{\partial\bar{\mu}}\right|_{\mu=\mu^*} = 0. \tag{3.56}$$

Since the attractor point has no flat directions one can obtain a unique solution for $\delta\mu$. Substituting the solution back into F given by (3.54) and using (3.56) it is easy to see that to $\mathcal{O}(Q^0,P^0)$ in charges the entropy is given by

$$F = \lambda^2 F^{(2)}(\mu^*,\bar{\mu}^*) + \frac{1}{2}\mathcal{F}(\mu^*,\bar{\mu}^*) + \mathcal{O}(1/\lambda^2). \tag{3.57}$$

Thus to evaluate the correction to entropy due to the Gauss-Bonnet term to $\mathcal{O}(Q^0,P^0)$ in charges it is sufficient to evaluate its coefficient at the attractor values of the moduli.

STU model. To evaluate the correction to the entropy including the Gauss-Bonnet term to $\mathcal{O}(Q^0,P^0)$ in charges we substitute the values of the vector multiplet at the attractor point given in (3.27), (3.31) and (3.34) into the coefficient of the Gauss-Bonnet term and using (3.57). We obtain

$$\begin{aligned} F &= \pi\sqrt{Q^2P^2-(Q\cdot P)^2} \\ &\quad - \ln\left[\vartheta_2(\tau)^4\vartheta_2(-\bar{\tau})^4(\tau_2)^2\right]|_* - \ln\left[\vartheta_2(y^+)^4\vartheta_2(-\bar{y}^+)^4(\tau_2)^2\right]|_* \\ &\quad - \ln\left[\vartheta_2(-1/y^-)^4\vartheta_2(1/\bar{y}^-)^4((-1/y^-)_2)^2\right]|_* + \mathcal{O}(1/Q^0,1/P^0). \end{aligned} \tag{3.58}$$

Here the subscript $*$ refers to the fact that we have substitute the attractor values for the moduli given in (3.27), (3.31) and (3.34). Note that using a modular transformation we have written the $\vartheta_4(y^-)$ in terms of $\vartheta_2(-1/y^-)$ in the coefficient of the Gauss-Bonnet term.

From the expressions for the moduli in (3.27), (3.31) and (3.34) and the fact that the leading term in the entropy is invariant under triality, it is easy to see that the corrected entropy given in (3.58) is invariant under the following triality symmetries.

$$\mathcal{T}_1 : W_1 \leftrightarrow \tilde{N}_2, \qquad ; W_2 \leftrightarrow -\tilde{N}_1. \tag{3.59}$$

Under this exchange of charges the attractor value of the moduli transform as

$$\tau_* \leftrightarrow y_*^+, \quad y_*^- \text{ invariant.} \tag{3.60}$$

Similarly under the exchange of the charges

$$\mathcal{T}_2 : W_1 \leftrightarrow N_1, \qquad ; \tilde{W}_1 \leftrightarrow \tilde{N}_1, \tag{3.61}$$

the attractor values of the moduli transform as

$$y_*^+ \leftrightarrow -\frac{1}{y_*^+}, \qquad \tau \text{ invariant.} \tag{3.62}$$

Finally under the exchange of charges

$$\mathcal{T}_3 : \tilde{N}_2 \leftrightarrow N_1 \qquad ; \tilde{W}_1 \rightarrow -W_2, \tag{3.63}$$

the attractor values of the moduli transform as

$$\tau_* \rightarrow -\frac{1}{y_*^-}, \qquad y^+ \text{ invariant.} \tag{3.64}$$

In the next section we write down a partition function which for dyons in the STU model which captures the subleading coefficient obtained by considering the Gauss-Bonnet term. The partition function manifestly has all the symmetries of the model.

The FHSV model. To obtain the entropy of the dyonic black hole in the FHSV model with the Gauss-Bonnet term to $\mathcal{O}(Q^0, P^0)$ in charges we substitute the attractor values of the the moduli given in (3.47), (3.48) in (3.57). The coefficient of the Gauss-Bonnet term is obtained from (2.11). Performing this we obtain

$$\begin{aligned} F &= \pi\sqrt{Q^2P^2 - (Q\cdot P)^2} - \ln\left(\eta(2\tau)^{12}\eta(-2\bar{\tau})^{12}\tau_2^6\right)|_* \\ &\quad -\frac{1}{2}\ln\left(\Phi_{\rm BE}(y)\Phi_{\rm BE}(-\bar{y})(2y_2^+y_2^- - \vec{y}^2)^4\right)|_* + \mathcal{O}(1/Q^2, 1/P^2). \end{aligned} \tag{3.65}$$

In this model there is a scaling limit of the charges in which the contribution of entropy from the the T-moduli becomes negligible compared that of the axion-dilaton moduli. Let us consider the following scaling of the charges

$$\begin{aligned} &Q^I \rightarrow \lambda Q^I, \qquad P^I \rightarrow \lambda' P^I \quad \text{with } \lambda \gg 1,\ \lambda' \gg 1 \\ &\text{and } \lambda \gg \lambda'. \end{aligned} \tag{3.66}$$

From the solution of the T-moduli given in (3.47), (3.48), it is easy to see that in this scaling limit the T-moduli are of order 1, $\mathcal{O}(\lambda^0, \lambda'^0)$. The axion dilaton moduli on the other hand scales as

$$\tau_1 \to \frac{\lambda}{\lambda'}\tau_1, \qquad \tau_2 \to \frac{\lambda}{\lambda'}\tau_2. \tag{3.67}$$

Thus in (3.65) we can neglect the contribution of the T-moduli and retain the leading contribution from the axion-dilaton moduli. Therefore (3.65) reduces to

$$\begin{aligned} F &= \pi\sqrt{Q^2P^2 - (Q\cdot P)^2} + 2\pi\tau_2|_* - 6\ln\tau_2|_* + \mathcal{O}(Q^0, P^0) \\ &= \pi\sqrt{Q^2P^2 - (Q\cdot P)^2} + 2\pi\frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2} - 6\ln\left(\frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}\right) \\ &\quad + \mathcal{O}(Q^0, P^0). \end{aligned} \tag{3.68}$$

In section 4. we will write down an approximate dyon partition function involving a Siegel modular form of weight 4 which captures the degeneracy of the dyons in the FHSV model in the limit when the electric charges are much larger than the magnetic charges as in (3.66). Note that in this limit though we loose most of the details of the correction to the entropy from the Gauss-Bonnet term, the information of the weight of the $\mathrm{SL}(2,\mathbb{Z})$ modular form which captures the axion-dilaton dependence is present in the coefficient of the logarithm in (3.68).

4. Partition function for the STU model

In this section we propose a partition function for the dyons of the STU model. The partition function is given in terms of product of three Siegel modular forms of weight zero. For the purposes of extracting the degeneracy from the partition function we need to define the charge bilinears which occur in the Fourier expansion of the partition function. From the consideration of the entropy function in the pervious section we have seen that that the following charge bilinears characterize the black hole entropy of the dyons to the next leading order. There are three sets of charge bilinears, each set is invariant with respect to any two of the three $\Gamma(2)$ symmetries of the STU model. The T-duality and U-duality invariants are given by

$$M_{\sigma^1} = \frac{1}{3}Q\cdot Q, \qquad M_{\rho^1} = \frac{1}{3}P\cdot P, \qquad M_{v^2} = \frac{1}{3}Q\cdot P.$$

The S-duality invariants are given by following anti-symmetric second rank tensor T^{ij} of $\mathrm{SO}(2,2)$

$$T^{ij} = Q^iP^j - Q^jP^i \tag{4.1}$$

where $i, j = 1, \cdots 4$. From the above 6 components of the S-duality invariant the charge bilinears we can project into the self-dual or the anti-self dual combinations which are further invariant under either T-duality or the U-duality symmetries. This projection is done by the 't Hooft symbols of $\mathrm{SO}(2,2)$. They are given by

$$\begin{aligned} M_{\sigma^2} &= \frac{1}{3}\eta_{-;ij}T^{ij}, \qquad M_{\rho^2} = \frac{1}{3}\eta_{+;ij}T^{ij}, \qquad M_{v^2} = \frac{1}{3}\eta_{1;ij}T^{ij}, \\ M_{\sigma^3} &= \frac{1}{3}\tilde{\eta}_{-;ij}T^{ij}, \qquad M_{\rho^3} = \frac{1}{3}\tilde{\eta}_{+;ij}T^{ij}, \qquad M_{v^3} = -\frac{1}{3}\tilde{\eta}_{1;ij}T^{ij}. \end{aligned} \tag{4.2}$$

Here $\eta_{a,ij}$ and $\tilde{\eta}_{a,ij}$ are the 't Hooft symbols which decomposes the second rank anti-symmetric tensor T^{ij} of SO(2, 2) and as $(\mathbf{3}, 0)$ and $(0, \mathbf{3})$ of SU(1, 1) × SU(1, 1). $a, = 1, \cdots 3$. These 't Hooft symbols are defined in appendix A.

We now propose the partition function for this model: The partition function involves the product of inverses of three Sp(2, $\mathbb{Z}$) modular forms of weight zero and is given by

$$
\begin{aligned}
d(M_\rho, M_\sigma, M_v) &= \prod_{\alpha=1}^{3} I_\alpha(M_{\rho^\alpha}, M_{\sigma^\alpha}, M_{v^\alpha}), \\
I_\alpha(M_{\rho^\alpha}, M_{\sigma^\alpha}, M_{v^\alpha}) &= \frac{K}{4} \exp{(i\pi M_v)} \int_{\mathcal{C}^\alpha} d\tilde{\rho}^\alpha d\tilde{\sigma}^\alpha d\tilde{v}^\alpha \frac{1}{\widetilde{\Phi}_0(\tilde{\rho}^\alpha, \tilde{\sigma}^\alpha, \tilde{v}^\alpha)} \\
&\quad \times \exp\left(-2\pi i \left[\frac{M_{\rho^\alpha}}{2}\tilde{\rho}^\alpha + \frac{M_{\sigma^\alpha}}{2}\tilde{\sigma}^\alpha + M_{v^\alpha}\tilde{v}\right]\right).
\end{aligned}
\tag{4.3}
$$

The Sp(2, $\mathbb{Z}$) modular form $\widetilde{\Phi}_0(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ which occurs in the partition function is given by

$$
\widetilde{\Phi}_0 = \widetilde{\Phi}_2 \sqrt{\frac{\widetilde{\Phi}'_2}{\widetilde{\Phi}_6}}. \tag{4.4}
$$

Here $\widetilde{\Phi}_6$ is the Siegel modular form of weight 6 under the subgroup $\tilde{G}$ of Sp(2, $\mathbb{Z}$) which captures the degeneracy of dyons in the $\mathbb{Z}_2$ CHL model [6, 7] $\widetilde{\Phi}_2$ is the Siegel modular form of weight 2 under the same subgroup $\tilde{G}$ of Sp(2, $\mathbb{Z}$) which captures the degeneracy of dyons in the $\mathbb{Z}_2$ orbifold of type IIB theory [10]. Finally $\widetilde{\Phi}'_2$ is the modular form of weight 2 related to $\widetilde{\Phi}$ by

$$
\widetilde{\Phi}'_2(\tilde{\rho}, \tilde{;}\sigma, \tilde{v}) = \widetilde{\Phi}_2\left(\frac{\tilde{\sigma}}{2}, 2\tilde{\rho}, \tilde{v}\right). \tag{4.5}
$$

From property 1 of $\widetilde{\Phi}'_2$ listed in the appendix C, we see that $\widetilde{\Phi}'_2$ is also a Siegel modular form of weight 2 under the subgroup $\tilde{G}$ of Sp(2, $\mathbb{Z}$). It is clear from the definition of $\widetilde{\Phi}_0$ that it is a modular form of weight 0, under the same subgroup $\tilde{G}$ of Sp(2, $\mathbb{Z}$). All the modular forms given in (4.4) are explicitly constructed in the appendix C in their product representation from appropriate threshold integrals. In (4.3) the three dimensional contour $\mathcal{C}^\alpha$ is given by

$$
\begin{gathered}
\mathrm{Im}\,\tilde{\rho}^\alpha = M_1^\alpha,\ \mathrm{Im}\,\tilde{\sigma}^\alpha = M_2^\alpha,\ \mathrm{Im}\,\tilde{v}^\alpha = M_3^\alpha, \\
0 \le \mathrm{Re}\,\tilde{\rho}^\alpha \le 2,\ 0 \le \mathrm{Re}\,\tilde{\sigma}^\alpha \le 2,\ 0 \le \mathrm{Re}\,\tilde{v}^\alpha \le 1.
\end{gathered}
\tag{4.6}
$$

The constant $K = -2^{-10}$. From the Fourier expansion of the partition function it can be shown that the degeneracy formula in (4.3) is valid with

$$
M_{\rho^\alpha} \in \mathbb{Z}, \quad M_{\sigma^\alpha} \in \mathbb{Z}, \quad M_{v^\alpha} \in \mathbb{Z}. \tag{4.7}
$$

We now list the properties of $\widetilde{\Phi}_0$ which are proved in the appendix C.

Properties of $\widetilde{\Phi}_0$.

1. $\widetilde{\Phi}_0$ is a modular form of weight 0 under the subgroup $\tilde{G}$ os Sp(2, $\mathbb{Z}$).

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2. $\widetilde{\Phi}_0(\tilde{\rho}, \tilde{\sigma}, v)$ is an analytic function with second order zeros at

$$
\begin{aligned}
& n_2(\tilde{\sigma}, \tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho} m_1 + m_2 = 0, \\
& \text{for } m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4}, \; m_1 \in 2\mathbb{Z}, n_1 \in \mathbb{Z}, b \in 2\mathbb{Z}+1, m_2, n_2 \in \mathbb{Z}.
\end{aligned}
\tag{4.8}
$$

It has second order poles at

$$
\begin{aligned}
& n_2(\tilde{\sigma}, \tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho} m_1 + m_2 = 0, \\
& \text{for } m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4}, \; m_1 \in 2\mathbb{Z}+1, n_1 \in \mathbb{Z}, b \in 2\mathbb{Z}+1, m_2, n_2 \in \mathbb{Z}.
\end{aligned}
\tag{4.9}
$$

and first order poles at

$$
\begin{aligned}
& n_2(\tilde{\sigma}, \tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho} m_1 + m_2 = 0, \\
& \text{for } m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4}, \; m_1 \in 2\mathbb{Z}, n_1 \in \mathbb{Z} + \frac{1}{2}, b \in 2\mathbb{Z}+1, m_2, n_2 \in \mathbb{Z}.
\end{aligned}
\tag{4.10}
$$

3. $\widetilde{\Phi}_0$ is invariant under the following $\mathrm{Sp}(2,\mathbb{Z})$ transformations

$$\Phi_0(\tilde{\rho}', \tilde{\sigma}', \tilde{v}') = \Phi_0(\tilde{\rho}, \tilde{\sigma}, \tilde{v}), \tag{4.11}$$

$$\text{for } \tilde{\rho}' = \tilde{\rho} + 4\tilde{\sigma} + 4\tilde{v} - 2, \quad \tilde{\sigma}' = \tilde{\sigma}, \quad \tilde{v}' = 2\tilde{\sigma} + \tilde{v}, \tag{4.12}$$

$$\text{and } \tilde{\rho}' = \tilde{\rho}, \quad \tilde{\sigma}' = 4\tilde{\rho} + \tilde{\sigma} + 4\tilde{v} - 2, \quad \tilde{v}' = 2\tilde{\sigma} + \tilde{v}. \tag{4.13}$$

This property is due to the fact that $\widetilde{\Phi}_0$ is invariant under the subgroup of $\tilde{H}$ defined in (C.7). The transformation (4.12) corresponds to the choice $a = -1, b = 2, c = 0, d = -1$ in (C.8) and (4.13) corresponds to the choice $a = 1, b = 0, c = -2, d = 1$ in (C.8). The above $\mathrm{Sp}(2,\mathbb{Z})$ transformations are essentially the generators of $\Gamma(2)$. Note that $\widetilde{\Phi}_0$ is invariant under the subgroup (C.7) which is $\Gamma_0(2)$ which contains the group $\Gamma(2)$.

4. The expansion of $1/\widetilde{\Phi}_0$ in terms of Fourier coefficients given by

$$\frac{K}{\widetilde{\Phi}_0(\tilde{\rho}, \tilde{\sigma}, \tilde{v})} = \sum_{\substack{m,n,p \\ m \geq -1/2, n \geq 1/2}} e^{2\pi i (m\tilde{\rho} + n\tilde{\sigma} + p\tilde{v})} g(m,n,p), \tag{4.14}$$

with $g(m,n,p)$ being integers, $p \in \mathbb{Z}$ while $m, n \in \mathbb{Z}/2$

5. For small $\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}$ $\widetilde{\Phi}_0$ factorizes as

$$\widetilde{\Phi}_0(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = 4\pi^2 (2v - \rho - \sigma)^2 v^2 f^{(0)}(\rho) f^{(0)}(\sigma) + \mathcal{O}(v^2), \tag{4.15}$$

where

$$f^{(0)}(\rho) = \vartheta_2(\rho)^4. \tag{4.16}$$

The variables (ρ, σ, v) and $(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$ are related by the $\mathrm{Sp}(2,\mathbb{Z})$ transformation

$$\rho = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2}{\tilde{\sigma}}, \qquad \sigma = \frac{\tilde{\rho}\tilde{\sigma} - (\tilde{v} - 1)^2}{\tilde{\sigma}}, \qquad v = \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}}{\tilde{\sigma}}, \tag{4.17}$$

or the inverse relations

$$\tilde{\rho} = \frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \qquad \tilde{\sigma} = \frac{1}{2v - \rho - \sigma}, \qquad \tilde{v} = \frac{v - \rho}{2v - \rho - \sigma}. \tag{4.18}$$

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We now perform several check on the proposal (4.3), we show that it has all the duality symmetries of the STU model and then show that for large charges the statistical entropy obtained from the partition function reproduces the degeneracy of dyonic black holes in the STU model to the first subleading order in charges.

4.1 Consistency checks

In this section we will subject our proposal given in (4.3) to various consistency checks. We first verify that the integrand in (4.3) has STU triality symmetry and then show that it has $\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$ symmetry.

STU triality symmetry. From the definitions of the charge bilinears it is easy to show the following transformation properties: Under

$$\mathcal{T}_1 : \; W_1 \leftrightarrow \tilde{N}_2, \qquad W_2 \leftrightarrow -\tilde{N}_1, \tag{4.19}$$

the following charge bilinears transform as

$$M_{\rho^1} \leftrightarrow M_{\rho^2}, \qquad M_{\sigma^1} \leftrightarrow M_{\sigma^1}, \quad M_{v^1} \leftrightarrow M_{v^2}, \tag{4.20}$$

while the charge bilinears $M_{\rho^3}, M_{\sigma^3}, M_{v^3}$ remains invariant. It is now easy to see that under this transformation $\mathcal{T}_1$the integrand in the partition function (4.3) is manifestly invariant. Similarly under

$$\mathcal{T}_2 : \; W_1 \leftrightarrow N_1, \qquad \tilde{W}_1 \leftrightarrow \tilde{N}_1, \tag{4.21}$$

that is the exchange of momentum and winding on the first circle, the following charge bilinears transform as

$$M_{\rho^2} \leftrightarrow M_{\rho^3}, \quad M_{\sigma^2} \leftrightarrow M_{\sigma^3}, \quad M_{v^2} \rightarrow M_{v^3}, \tag{4.22}$$

while the charge bilinears $M_{\rho^1}, M_{\sigma^1}, M_{v^1}$ remains invariant. Thus it is easy to see that the integrand in the partition function (4.3) is manifestly invariant under the transformation $\mathcal{T}_2$. Finally the transformation

$$\mathcal{T}_3 : \; \tilde{N}_2 \leftrightarrow N_1, \qquad \tilde{W}_1 \leftrightarrow -W_2, \tag{4.23}$$

the following charge bilinears transform as

$$M_{\rho^1} \leftrightarrow M_{\rho^3}, \quad M_{\sigma^1} \leftrightarrow M_{\sigma^3}, \quad M_{v^1} \rightarrow M_{v^3}, \tag{4.24}$$

while the charge bilinears $M_{\rho^2}, M_{\sigma^2}, M_{v^2}$ remains invariant. Thus it is easy to see that the integrand in the partition function (4.3) is manifestly invariant under the transformation $\mathcal{T}_3$. Therefore ignoring considerations of the contour $\mathcal{C}^\alpha$ the degeneracy is invariant under the triality symmetries $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$.

$\boldsymbol{\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U}$ symmetry. We first look at $\Gamma(2)_S$ action, which acts on the electric and magnetic charges as

$$Q \to Q' = aQ + bP, \quad P \to P' = cQ + dP, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2). \tag{4.25}$$

The generators of $\Gamma(2)$ are given by the following matrices

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{4.26}$$

Using (4.25) with the generator M_∞ we find that the T-duality invariants $M_{\rho^1}, M_{\sigma^1}, M_{v^1}$ transform as

$$\begin{pmatrix} M'_{\sigma^1} \\ M'_{\rho^1} \\ M'_{v^1} \end{pmatrix} = \begin{pmatrix} 1 & 4 & -4 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} M_{\sigma^1} \\ M_{\rho^1} \\ M_{v^1} \end{pmatrix}. \tag{4.27}$$

Now following [6] let us define

$$\tilde{\Omega}^{1'} \equiv \begin{pmatrix} \tilde{\rho}^{1'} & \tilde{v}^{1'} \\ \tilde{v}^{1'} & \tilde{\sigma}^{1\prime} \end{pmatrix} = (\tilde{A}\tilde{\Omega}^1 + \tilde{B})(\tilde{C}\tilde{\Omega}^1 + \tilde{D})^{-1}, \quad \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} a & -b & b & 0 \\ -c & d & 0 & c \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix}. \tag{4.28}$$

For the matrix M_∞ we obtain

$$\begin{aligned} \tilde{\rho}^{1'} &= \tilde{\rho}^1 + 4\tilde{\sigma}^1 + 4\tilde{v}^1 - 2, \\ \tilde{\sigma}^{1'} &= \tilde{\sigma}^1, \\ \tilde{v}^{1'} &= 2\tilde{\sigma}^1 + \tilde{v}^1. \end{aligned} \tag{4.29}$$

Using the transformations (4.27) and (4.29) we obtain the following transformations

$$\begin{aligned} &\exp\left[2\pi i\left(\frac{M'_{\rho^1}}{2}\rho^{1'} + \frac{M'_{\sigma^1}}{2}\sigma^{1'} + M'_{v^1}v^{1'}\right)\right] \\ &\qquad \to \exp\left[2\pi i\left(\frac{M_{\rho^1}}{2}\rho^1 + \frac{M_{\sigma^1}}{2}\sigma^1 + M_{v^1}v^1\right)\right]\exp\left(2\pi i M_{\rho^1}\right), \\ &\exp\left(i\pi M'_{v^1}\right) \to \exp\left(-2\pi i M_{\rho^1}\right)\exp\left(i\pi M_{v^1}\right). \end{aligned} \tag{4.30}$$

Furthermore from (4.29) and (4.11) we see that

$$\widetilde{\Phi}_0(\tilde{\rho}^{1'}, \tilde{\sigma}^{1'}, \tilde{v}^{1'}) = \widetilde{\Phi}_0(\tilde{\rho}^1, \tilde{\sigma}^1, \tilde{v}^1). \tag{4.31}$$

Finally one can show that

$$d\tilde{\rho}^{1'}d\tilde{\sigma}^{1'}d\tilde{v}^{1'} = d\tilde{\rho}^1 d\tilde{\sigma}^1 d\tilde{v}^1. \tag{4.32}$$

Combining (4.30), (4.32) and (4.31) we see that the the integrand in I_1 is invariant under M_∞. Since the transformations in (4.27) are S-duality transformations they leave the leave the the remaining charge bilinears in (4.2) invariant. Therefore the integrand in the partition function (4.3) remains invariant under the S-duality action of M_∞. Using the same argument one can show that the integrand in (4.3) is invariant under the S-duality action of M_1 under which the charge bilinears transform as

$$\begin{pmatrix} M'_{\sigma^1} \\ M'_{\rho^1} \\ M'_{v^1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & -4 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_{\sigma^1} \\ M_{\rho^1} \\ M_{v^1} \end{pmatrix}, \tag{4.33}$$

and the $\mathrm{Sp}(2,\mathbb{Z})$ variables transform as the equations in the third line of (4.11). We can thus conclude that the integrand is invariant under $\Gamma(2)_S$. In fact since the form $\widetilde{\Phi}_0$ is invariant under $\Gamma_0(2)$ (see property 1 of $\widetilde{\Phi}_0$ in appendix C.), the partition function is invariant under this larger symmetry.

The action of the two generators M_∞ and M_1 of $\Gamma(2)_T$ on the charge bilinears $M_{\rho^2}, M_{\sigma^2}, M_{v^2}$ is given by the the same equations (4.27) and (4.33) with the 1 replaced by 2. To show that the integrand is invariant under $\Gamma(2)_T$ is is sufficient to use the fact that the integrand is invariant under the triality symmetry $\mathcal{T}_1$. From the action of $\mathcal{T}_1$ on the charge bilinears given in (4.19), (4.20) we have the following relation between the action of $\Gamma(2)_T$ and $\Gamma(2)_S$

$$\Gamma(2)_T = \mathcal{T}_1 \Gamma(2)_S \mathcal{T}_1. \tag{4.34}$$

Since the integrand is invariant under $\Gamma(2)_S$ and the triality symmetry $\mathcal{T}_1$ we see that the $\Gamma(2)_S$ is a symmetry of the integrand. Finally the action of the two generators M_∞ and M_1 of $\Gamma(2)_U$ on the charge bilinears $M_{\rho^2}, M_{\sigma^2}, M_{v^2}$ is given by the the same equations (4.27) and (4.33) with the 1 replaced by 3. Using (4.23) and (4.24) we have

$$\Gamma(2)_U = \mathcal{T}_3 \Gamma(2)_S \mathcal{T}_3. \tag{4.35}$$

Since the integrand is invariant under $\Gamma(2)_S$ and the triality symmetry $\mathcal{T}_3$ we see that the $\Gamma(2)_S$ is a symmetry of the integrand. Thus we conclude that ignoring considerations of the contour the degeneracy given by (4.3) is invariant under $\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$.

Integrality of the $\boldsymbol{d(M_\rho, M_\sigma, M_v)}$. From property (4.14) of $\widetilde{\Phi}_0$ that Fourier coefficients $d(M_\rho, M_\sigma, M_v)$ are all integers. From our definition of the charge bilinears in (4.2) and from the moding of the Fourier expansions in (4.14) we see that

$$M_{\rho^\alpha} = \mathbb{Z}, \qquad M_{\sigma^\alpha} = \mathbb{Z}, \qquad M_{v^\alpha} = \mathbb{Z}. \tag{4.36}$$

Thus the class of dyons the partition function given in (4.3) is applicable has the above quantization conditions.

4.2 Statistical entropy of the STU model

In this section we obtain the asymptotic degeneracies for dyons for large charges $M_{\rho^\alpha}, M_{\sigma^\alpha}, M_{v^\alpha} \gg 0$ with $Q^2P^2 - (Q\cdot P)^2 \gg 0$. We begin with the formula (4.3) for the degeneracy of dyons in the STU model. The degeneracy is obtained as a product of three integrals given by

$$I_\alpha(M_{\rho^\alpha}, M_{\sigma^\alpha}, M_{v^\alpha}) = \frac{K}{4}\exp\left(i\pi M_{v^\alpha}\right)\int_{\mathcal{C}^\alpha} d\tilde{\rho}^\alpha d\tilde{\sigma}^\alpha d\tilde{v}^\alpha \frac{1}{\widetilde{\Phi}_0(\tilde{\rho}^\alpha, \tilde{\sigma}^\alpha, \tilde{v}^\alpha)} \times \exp\left[-2\pi i\left(\frac{M_{\rho^\alpha}}{2}\tilde{\rho}^\alpha + \frac{M_{\sigma^\alpha}}{2}\tilde{\sigma}^\alpha + M_{v^\alpha}\tilde{v}\right)\right]. \tag{4.37}$$

This formula is identical in form to equation (3.29) of [9] with the substitution $Q_e^2 \to M_{\sigma^\alpha},\ Q_m^2 \to M_{\rho^\alpha},\ Q_e \cdot Q_m \to M_{v^\alpha}$. Following [9] one can show that the dominant contribution to this integral comes form the residue at the pole at

$$\tilde{\rho}^\alpha\tilde{\sigma}^\alpha - (\tilde{v}^\alpha)^2 + \tilde{v}^\alpha = 0. \tag{4.38}$$

The behavior of $\widetilde{\Phi}_0$ near this zero is given by (4.15), is identical to the corresponding relation (4.17) in [9] with $k \to 0$ and $f^{(k)}(\rho) \to f^{(0)}(\rho)$ given in (4.16). Thus following an analysis identical to that in [9] we can conclude that for large charges the contribution to the statistical entropy form the integral I_α defined as the log of the contribution of the degeneracy $I_\alpha(M_{\rho^\alpha}, M_{\sigma^\alpha}, M_{v^\alpha})$ is obtained by minimizing the statistical entropy function

$$-\tilde{\Gamma}_B^\alpha(\vec{\gamma}^\alpha) = \frac{\pi}{2\gamma_2^\alpha}\left(M_{\sigma^\alpha} + 2\gamma_1^\alpha M_v + \tau^\gamma\bar{\tau}^\gamma M_{\rho^\alpha}\right) - \ln\left[\vartheta_2(\gamma^\alpha)^4\vartheta_2(-\bar{\gamma}^\alpha)^4(2\gamma_2^\alpha)^2\right] + \text{constant} + \mathcal{O}(1/Q^2), \tag{4.39}$$

with respect to the real and imaginary parts of γ^α. To order $\mathcal{O}(1/Q^2)$ it is sufficient to obtain the value of τ^α at the minimum by minimizing the $\mathcal{O}(Q^2)$ term in (4.39). This is given by

$$\begin{aligned}\gamma_1^\alpha|_* &= -\frac{M_{v^\alpha}}{M_{\rho^\alpha}},\\ \gamma_2^\alpha|_* &= \frac{\sqrt{M_{\rho^\alpha}M_{\sigma^\alpha} - M_{v^\alpha}^2}}{M_{\rho^\alpha}},\\ &= \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{3M_{\rho^\alpha}}.\end{aligned} \tag{4.40}$$

We can substituting these values in the statistical entropy function (4.39) to obtain the value of the statistical entropy from I_α to $\mathcal{O}(1/Q^2)$, this is given by

$$\begin{aligned}-\tilde{\Gamma}_B(\vec{\gamma}^\alpha)|_* &= \pi\sqrt{M_{\rho^\alpha}M_{\sigma^\alpha} - M_{v^\alpha}^2}\\ &\quad - \ln\left[\vartheta_2(\gamma^\alpha)^4\vartheta_2(-\bar{\gamma}^\alpha)^4(2\gamma_2^\alpha)^2\right]\Big|_* + \text{constant} + \mathcal{O}(1/Q^2),\\ &= \frac{\pi}{3}\sqrt{Q^2P^2 - (Q\cdot P)^2}\\ &\quad - \ln\left[\vartheta_2(\gamma^\alpha)^4\vartheta_2(-\bar{\gamma}^\alpha)^4(2\gamma_2^\alpha)^2\right]\Big|_* + \text{constant} + \mathcal{O}(1/Q^2).\end{aligned} \tag{4.41}$$

The total statistical entropy is then given by

$$-\tilde{\Gamma}_B(\vec{\gamma}^1, \vec{\gamma}^2, \vec{\gamma}^3) = -\sum_{\alpha=1}^{3}\tilde{\Gamma}_B(\vec{\gamma}^\alpha). \tag{4.42}$$

From comparing (3.34), (3.25), (3.32) to the minimum values in (4.40) and using the definition of the charge bilinears in (4.1) and (4.2) it is seen that we obtain the following equations

$$\gamma^1|_* = \tau|_*, \qquad \gamma^2|_* = y^+|_*, \qquad \gamma^3|_* = -\frac{1}{y^-}|_*. \tag{4.43}$$

From (4.42) and (4.41) we see that to $\mathcal{O}(1/Q^2)$, the total statistical entropy is given by

$$\begin{aligned} -\tilde{\Gamma}_B(\vec{\gamma}^1,\vec{\gamma}^2,\vec{\gamma}^3)|_* &= \pi\sqrt{Q^2P^2-(Q\cdot P)^2} \\ &\quad -\sum_{\alpha=1}^{3}\ln\left[\vartheta_2(\gamma^\alpha)^4\vartheta_2(-\bar{\gamma}^\alpha)^4(2\gamma_2^\alpha)^2\right]\big|_* + \text{constant} + \mathcal{O}(1/Q^2). \end{aligned} \tag{4.44}$$

Using (4.43) and (4.44) we see that the statistical entropy coincides with the entropy of the black hole to the next leading order.

5. Approximate partition function for the FHSV model

The subleading corrections to the entropy from the the coefficient of the Gauss-Bonnet in the FHSV model depends on the attractor values of the T-moduli through the Bocherds-Enriques form (3.65). The complete dyon partition function should capture this dependence on the T-moduli. Here we will focus on the scaling limit (3.66) in which the electric charges are much larger than the magnetic charges and write down an approximate dyon partition function which captures the degeneracy in this limit.

$$d(Q,P) = \frac{K'}{2}\exp i\pi(Q\cdot P)\int_C d\tilde{\rho}d\tilde{\sigma}d\tilde{v}\frac{1}{\widetilde{\Phi}_4(\tilde{\rho},\tilde{\sigma},\tilde{v})}\exp\left[-i\pi\left(\tilde{\sigma}Q^2+\tilde{\rho}P^2+2\tilde{v}Q\cdot P\right)\right], \tag{5.1}$$

where $Q^2 \equiv Q\cdot Q$, $P^2 \equiv P\cdot P$, $\widetilde{\Phi}_4$ is a function to be specified, and C is a three real dimensional subspace of the three complex dimensional space labeled by $(\tilde{\rho},\tilde{\sigma},\tilde{v})$ given by

$$\begin{gathered} \operatorname{Im}\tilde{\rho} = M_1 \quad \operatorname{Im}\tilde{\sigma} = M_2, \qquad \operatorname{Im}\tilde{v} = M_3, \\ 0\le \operatorname{Re}\tilde{\rho}\le 1, \quad 0\le \operatorname{Re}\tilde{\sigma}\le 2, \quad 0\le \operatorname{Re}\tilde{v}\le 1. \end{gathered} \tag{5.2}$$

M_1, M_2, M_3 being fixed large positive numbers. The normalization constant in (5.1) is given by $K' = 2^{-6}$. The $\mathrm{Sp}(2,\mathbb{Z})$ modular form Φ_4 which occurs in (5.1) is by

$$\widetilde{\Phi}_4(\tilde{\rho},\tilde{\sigma},\tilde{v}) = \left(\widetilde{\Phi}_6(\tilde{\rho},\tilde{\sigma},\tilde{v})\widetilde{\Phi}_2(\tilde{\rho},\tilde{\sigma},\tilde{v})\right)^{\frac{1}{2}}, \tag{5.3}$$

where $\widetilde{\Phi}_6$ is the modular form of weight 6 which captures the degeneracy of dyons for the $N=2$ CHL orbifold discussed in [6, 7] Similarly $\widetilde{\Phi}_2$ is the modular form of $\mathrm{Sp}(2,\mathbb{Z})$ of weight 2 which capture degeneracy of dyons for the $\mathbb{Z}_2$ orbifold of type II theory constructed in [10]. From the definition of $\widetilde{\Phi}_4$ in (5.3) it is easily seen that it is a modular form of weight 4 under the subgroup $\tilde{G}$ of $\mathrm{Sp}(2,\mathbb{Z})$ defined in [6]. Thus we have

$$\widetilde{\Phi}_4\left(A\Omega+B)(C\Omega+D)^{-1}\right) = \det(C\Omega+D)^4\widetilde{\Phi}_4(\Omega), \qquad \begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\tilde{G}. \tag{5.4}$$

We now list some properties of $\widetilde{\Phi}_4$ which are discussed in the the appendix in detail.

Properties of $\widetilde{\Phi}_4$.

1. $\widetilde{\Phi}_4(\tilde\rho,\tilde\sigma,\tilde v)$ is an analytic function in $\tilde\rho,\tilde\sigma,\tilde v$ with second order zeros at

$$\begin{aligned}\big(n_2(\tilde\sigma\tilde\rho-\tilde v^2)+b\tilde v+n_1\tilde\sigma-\tilde\rho m_1+m_2\big)&=0, \qquad (5.5)\\ \text{for } m_1\in 2\mathbb{Z}, m_2,n_2\in\mathbb{Z}, b\in 2\mathbb{Z}+1, m_1n_1+m_2n_2+\frac{b^2}{4}&=\frac{1}{4}.\end{aligned}$$

   It has simple poles at

$$\begin{aligned}\big(n_2(\tilde\sigma\tilde\rho-\tilde v^2)+b\tilde v+n_1\tilde\sigma-\tilde\rho m_1+m_2\big)&=0, \qquad (5.6)\\ \text{for } m_1\in 2\mathbb{Z}+1, m_2,n_2\in\mathbb{Z}, n_1\in\mathbb{Z}, b\in 2\mathbb{Z}+1, m_1n_1+m_2n_2+\frac{b^2}{4}&=\frac{1}{4}.\end{aligned}$$

2. $\widetilde{\Phi}_4$ is invariant under the following $\mathrm{Sp}(2,\mathbb{Z})$ transformations

$$\widetilde{\Phi}_4(\tilde\rho',\tilde\sigma',\tilde v')=\widetilde{\Phi}_4(\tilde\rho,\tilde\sigma,\tilde v), \qquad (5.7)$$
$$\text{for } \tilde\rho'=\tilde\rho+4\tilde\sigma+4\tilde v-2, \quad \tilde\sigma'=\tilde\sigma, \quad \tilde v'=2\tilde\sigma+\tilde v, \qquad (5.8)$$
$$\text{and } \tilde\rho'=\tilde\rho, \quad \tilde\sigma'=4\tilde\rho+\tilde\sigma+4\tilde v-2, \quad \tilde v'=2\tilde\sigma+\tilde v. \qquad (5.9)$$

   This property is due to the fact that $\widetilde{\Phi}_4$ is invariant under the subgroup of $\tilde H$ defined in (C.7). The transformation (5.8) corresponds to the choice $a=-1, b=2, c=0, d=-1$ in (C.8) and (5.9) corresponds to the choice $a=1, b=0, c=-2, d=1$ in (C.8). The above $\mathrm{Sp}(2,\mathbb{Z})$ transformations essentially correspond to generators of $\Gamma(2)$, but $\widetilde{\Phi}_4$ is invariant under the subgroup (C.7) which is $\Gamma_0(2)$, this contains the group $\Gamma(2)$.

3. The expansion of $1/\widetilde{\Phi}_4$ in terms of Fourier coefficients given by

$$\frac{K'}{\widetilde{\Phi}_4(\tilde\rho,\tilde\sigma,\tilde v)}=\sum_{\substack{m,n,p\\ m\geq -1, n\geq -1/2}} e^{2\pi i(m\tilde\rho+n\tilde\sigma+p\tilde v)}g(m,n,p), \qquad (5.10)$$

   with $g(m,n,p)$ being integers. $m,p\in\mathbb{Z}$ while n runs over integer multiples of $1/2$

4. To determine the asymptotic properties of the partition function given in (5.1) for large charges we need the behavior of $\widetilde{\Phi}_4$ near $\tilde\sigma\tilde\rho-\tilde v^2+\tilde v=0$. Near this surface $\widetilde{\Phi}_4$ factorizes as

$$\widetilde{\Phi}_4(\tilde\rho,\tilde\sigma,\tilde v)=4\pi^2(2v-\rho-\sigma)^2v^2f(\rho)f(\sigma)+\mathcal{O}(v^4), \qquad (5.11)$$

   where

$$f^{(4)}(\rho)=\eta(2\rho)^{12}, \qquad (5.12)$$

   and the relationship between the variables ρ,σ,v and $\tilde\rho,\tilde\sigma,\tilde v$ is given by the $\mathrm{Sp}(2,\mathbb{Z})$ transformation (4.17) and (4.18).

Just as in the case of $\widetilde{\Phi}_0$ in the previous section, following the same logic and using the fact that $\widetilde{\Phi}_4$ is also invariant under the $\mathrm{Sp}(2,\mathbb{Z})$ transformations given in (5.8)and (5.9), we see that the integrand in (5.1) is invariant under $\Gamma(2)_S$ symmetry. This is what is expected of a partition function which aims to capture the dependence of the axion-dilaton moduli dependence of the subleading terms in the entropy function. In fact from property since $\widetilde{\Phi}_4$ is invariant under the subgroup (C.7), the partition function (5.1) is invariant under the larger group $\Gamma_0(2)$.

We shall now compute the statistical entropy from the approximate partition function for the FHSV model given in (5.1). We will see that the statistical entropy from this partition function captures the axion-dilaton moduli dependence of the entropy (3.65). To obtain the statistical entropy we will follow the method of [9]. of the dyons in the FHSV model. The value of this function at its extremum gives the statistical entropy, the logarithm of the degeneracy of states corresponding to the given set of charges. We see that the black hole entropy function evaluated in section 4 agrees precisely with the statistical entropy. Following [9, 1, 33] one can show that the dominant contribution to the above integral comes form the pole at

$$\tilde{\sigma}\tilde{\rho} - \tilde{v}^2 + \tilde{v} = 0. \tag{5.13}$$

The behavior of $\widetilde{\Phi}_4$ near this zero is given by (5.11), which is identical to the corresponding relation (4.17) with $k \to 4$ and $f^{(k)}(\rho) \to f(\rho)$. Thus following an analysis identical to that in [9] we can conclude that for large charges the statistical entropy $S_{\rm stat}(Q,P)$ defined as the logarithm of the degeneracy $d(Q,P)$ is obtained by minimizing the statistical entropy function

$$\begin{aligned} -\tilde{\Gamma}_B(\vec{\tau}) = & \frac{\pi}{2\tau_2}|Q+\tau P|^2 - \ln(f^{(4)}(\tau)) - \ln(f^{(4)}(-\bar{\tau})) - 6\ln(2\tau_2) + \text{constant} \\ & + \mathcal{O}(1/Q^2, 1/P^2), \end{aligned} \tag{5.14}$$

where

$$f^{(4)}(\tau) = \eta(2\tau)^{12}. \tag{5.15}$$

We can then obtain the statistical entropy by first minimizing the the function $-\tilde{\Gamma}_B$ with respect to the real and imaginary parts of τ and then evaluating the value of the statistical entropy function at this critical point. This gives

$$-\tilde{\Gamma}_B(\vec{\tau})|_* = \pi\sqrt{Q^2P^2-(Q\cdot P)^2} - \ln\left[f^{(4)}(\tau)f^{(4)}(-\bar{\tau})(2\tau_2)^6\right]|_*, \tag{5.16}$$

where

$$\tau_1|_* = -\frac{Q\cdot P}{P^2}, \qquad \tau_2|_* = \frac{\sqrt{Q^2P^2-(Q\cdot P)^2}}{P^2}. \tag{5.17}$$

Comparing (3.65) and (5.16) using (3.50) and (5.17) we see that the approximate partition function given in (5.1) captures the contribution of the axion-dilaton moduli in the subleading terms of the black hole entropy obtained from the Gauss-Bonnet term. Therefore in the scaling limit given in (3.66) where the T-moduli contribution to the entropy is subleading, the approximate partition function given in (5.1) agrees with the black hole entropy (3.68) ignoring $\mathcal{O}(Q^0, P^0)$ terms.

We note that the vector multiplet moduli space of the FHSV model factorizes into the axion-dilaton dependence and the T-moduli dependence. We also have seen that the partition function in (5.1) captures the subleading contribution to the dyon entropy form the axion-dilaton dependence of the Gauss-Bonnet term. Thus we can conclude that a product of $\widetilde{\Phi}_4$ with a suitable function should capture the degeneracy of dyons in the FHSV model. It will be interesting to determine this function using clues from the corrections to the entropy function given in (3.65) and the attractor values of the T-moduli given in (3.47) and (3.48).

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A. 't Hooft symbols for SO(2, 2)

We obtain the 't Hooft symbols for SO(2, 2) from the corresponding symbols for SO(4) given by [34] by the prescription: $\eta_{a\mu\nu} \to (i)^{\delta_{a1}-\delta_{a2}+\delta_{\mu 3}+\delta_{\nu 4}}\eta_{a\mu\nu}$ with an identical prescription for the anti-self dual 't Hooft symbols. This prescription takes care of the appropriate signature changes required in going from SO(4) to SO(2, 2). Using this prescription we obtain the following self dual 't Hooft symbols for SO(2, 2), $\eta_{a;ij}$ with $a = 1, 2, 3$ and $i, j = 1, \cdots 4$ are defined as

$$\begin{aligned} \eta_{1;23} &= -1, & \eta_{2;31} &= 1, & \eta_{3;12} &= 1, \\ \eta_{1;32} &= 1, & \eta_{2;13} &= -1, & \eta_{3;21} &= -1, \\ \eta_{1;41} &= 1, & \eta_{2;42} &= -1, & \eta_{3;43} &= +1, \\ \eta_{1;14} &= -1, & \eta_{2;24} &= 1, & \eta_{3;34} &= -1. \end{aligned} \tag{A.1}$$

All the remaining components vanish. The anti-self dual 't Hooft symbols for SO(2, 2,) $\tilde{\eta}_{a;ij}$ are defined as

$$\begin{aligned} \tilde{\eta}_{1;23} &= -1, & \tilde{\eta}_{2;31} &= 1, & \tilde{\eta}_{3;12} &= 1, \\ \tilde{\eta}_{1;32} &= 1, & \tilde{\eta}_{2;13} &= -1, & \tilde{\eta}_{3;21} &= -1, \\ \tilde{\eta}_{1;41} &= -1, & \tilde{\eta}_{2;42} &= 1, & \tilde{\eta}_{3;43} &= -1, \\ \tilde{\eta}_{1;14} &= 1, & \tilde{\eta}_{2;24} &= -1, & \eta_{3;34} &= 1. \end{aligned} \tag{A.2}$$

They satisfy the following identities:

$$\eta_{a;ij} = \frac{1}{2}\epsilon_{ijkl}\eta_a^{kl}, \qquad \tilde{\eta}_{a;ij} = -\frac{1}{2}\epsilon_{ijkl}\tilde{\eta}_a^{kl}, \tag{A.3}$$

$$\eta_{a;ij} = -\eta_{aji}, \qquad \tilde{\eta}_{a;ij} = -\tilde{\eta}_{a;ji}, \tag{A.4}$$

$$\eta_{a;ij}\eta_b^{ij} = 4n_{ab}, \qquad \tilde{\eta}_{a;ij}\tilde{\eta}_b^{ij} = 4n_{ab}, \tag{A.5}$$

$$\eta_{a;ij}\eta^a_{kl} = L_{ik}L_{jl} - L_{il}L_{jk} + \epsilon_{ijkl}, \tag{A.6}$$

$$\tilde{\eta}_{a;ij}\tilde{\eta}^a_{kl} = L_{ik}L_{jl} - L_{il}L_{jk} - \epsilon_{ijkl}. \tag{A.7}$$

Raising and lowering of i, j indices are performed using the metric $L_{ij} = \mathrm{Dia}(1, 1, -1, -1)$. The $S0(2, 1)$ metric n_{ab} is given by $n_{ab} = \mathrm{Dia}(-1, -1, 1)$ and the raising and lowering of a, b indices are performed by the metric n_{ab}. We define the following combination of the 't Hooft symbols as

$$\eta_{\pm ij} = \pm\eta_{2;ij} + \eta_{3;ij}, \qquad \tilde{\eta}_{\pm ij} = \pm\tilde{\eta}_{2;ij} + \tilde{\eta}_{3;ij}. \tag{A.8}$$

B. Attractor values for the T-moduli in the FHSV model

In this part of the appendix we provide the details of the calculations which leads to the attractor values of the T-moduli in the FHSV model given in (3.47) and (3.48). We start with the entropy function of the FHSV model given in (3.45), to determine the values of the T-moduli at the attractor point it is sufficient to focus on the first two terms of (3.45). The first two terms are identical except for the exchange of $\tau \leftrightarrow \bar{\tau}$ in the numerator. Our strategy for minimizing with respect to the T-moduli is similar to the one we followed in the case of the STU model. We will just focus on the first term of (3.45), this is is given by

$$F_{\mathrm{T}} = \frac{\pi}{2}\frac{|(Q + \tau P)\cdot w|^2}{\tau_2 Y}, \tag{B.1}$$

and minimize this term with respect to the T-moduli. We will see that the attractor values of the moduli are independent of the axion-dilator moduli τ. Therefore these values of the attractor moduli minimize the second term in (3.45) simultaneously. This is because the second term is the same as the first term with $\tau \to \bar{\tau}$. For convenience we introduce the variables

$$\begin{aligned}
&n_1 = (Q^{12} + Q^{10}) + \tau(P^{12} + P^{10}), \quad n_2 = (Q^{11} + Q^9) + \tau(P^{11} + P^9) \\
&w_1' = (Q^{12} - Q^{10}) + \tau(P^{12} - P^{10}), \quad w_2' = (Q^9 - Q^{11}) + \tau(P^9 - P^{11}), \\
&q^i = Q^i + \tau P^i, \quad i = 1, \cdots 8.
\end{aligned} \tag{B.2}$$

With these variables one can write (B.1) as

$$\begin{aligned}
F_{\mathrm{T}} &= -4\frac{|E|^2}{D} \\
\text{where } E &= -\vec{q}\cdot\vec{y} - w_1'\frac{\vec{y}^2}{4} - n_1 + \frac{y^+y^-}{2}w_1' + \frac{y^+}{\sqrt{2}}w_2' + \frac{y^-}{\sqrt{2}}n_2, \\
\text{and } D &= 2(y^+ - \bar{y}^+)(y^- - \bar{y}^-) - \sum_{i=1}^{8}(y^i - \bar{y}^i)^2,
\end{aligned} \tag{B.3}$$

where we have substituted for w in terms of y using (3.39) and used the variables given in (B.2). Minimizing with respect to y^+ we get the equation

$$\frac{y^+ w_1'}{2} + \frac{n_2}{\sqrt{2}} = 2\frac{E}{D}(y^+ - \bar{y}^+). \tag{B.4}$$

The above equation can be simplified by further substituting for E and D in terms of the y's, this results in

$$\bar{y}^+ = \frac{n_1 - \frac{y^- n_2}{\sqrt{2}}}{\frac{y^- w_1'}{2} + \frac{w_2'}{\sqrt{2}}} - \sum_{i=1}^{8} \frac{(y^i - \bar{y}^i)^2}{2(y - \bar{y})} + \frac{\vec{q}\cdot\vec{y} + \frac{\vec{y}^2}{4} w_1'}{\frac{y^- w_1'}{2} + \frac{w_2'}{\sqrt{2}}}. \tag{B.5}$$

Minimizing with respect to y^- and y^i we obtain the equations

$$\frac{y^- w_1'}{2} + \frac{w_2'}{\sqrt{2}} = 2\frac{E}{D}(y^- - \bar{y}^-), \tag{B.6}$$

$$q^i + \frac{w_1'}{2} y^i = 2\frac{E}{D}(y^i - \bar{y}^i). \tag{B.7}$$

From (B.4), (B.6) and (B.7) we can solve for y^+, y^- and y^- in terms of the ratio

$$R = 2\frac{E}{D}. \tag{B.8}$$

These are given by

$$\begin{aligned}
y^+ &= \frac{1}{(\frac{w_1'}{2} - R)(\frac{\bar{w}_1'}{2} - \bar{R}) - \bar{R}R} \left(-\frac{n_2}{2\sqrt{2}} \bar{w}_1' + \frac{n_2}{\sqrt{2}} \bar{R} + \frac{\bar{n}_2}{\sqrt{2}} R \right), \\
y^- &= \frac{1}{(\frac{w_1'}{2} - R)(\frac{\bar{w}_1'}{2} - \bar{R}) - \bar{R}R} \left(-\frac{w_2'}{2\sqrt{2}} \bar{w}_1' + \frac{w_2'}{\sqrt{2}} \bar{R} + \frac{\bar{w}_2'}{\sqrt{2}} R \right) \\
y^i &= \frac{1}{(\frac{w_1'}{2} - R)(\frac{\bar{w}_1'}{2} - \bar{R}) - \bar{R}R} \left(-\frac{q^i}{2} \bar{w}_1' + q^i \bar{R} + \bar{q}^i R \right).
\end{aligned} \tag{B.9}$$

From the above equations it is possible to evaluate the following ratios which are independent of R.

$$\frac{y^- - \bar{y}^-}{y^+ - \bar{y}^+} = \frac{\bar{w}_2' w_1' - w_2' \bar{w}_1'}{\bar{n}_2 w_1' - n_2 \bar{w}_1'}, \tag{B.10}$$

$$\frac{y^i - \bar{y}^i}{y^+ - \bar{y}^+} = \sqrt{2}\frac{\bar{q}^i w_1' - q^i \bar{w}_1'}{\bar{n}_2 w_1' - n_2 \bar{w}_1'}. \tag{B.11}$$

It is easy to see that the ratios on the right hand side of the above equations are independent of the axion-dilaton moduli τ on substituting the definitions (B.2). Now from (B.4) and (B.6) and (B.10) we can obtain the value of y^- in terms of y^+ this is given by

$$\begin{aligned}
y^- &= y^+ \frac{\bar{w}_2 w_1' - w_2 \bar{w}_1'}{\bar{n}_2 w_1' - n_2 \bar{w}_1'} + \sqrt{2}\frac{n_2 \bar{w}_2' - w_2' \bar{n}_2}{\bar{n}_2 w_1' - n_2 \bar{w}_1'}, \\
&= y^+ \frac{W_2 \tilde{W}_1 - \tilde{W}_2 W_1}{N_2 \tilde{W}_1 - \tilde{N}_2 W_1} + \sqrt{2}\frac{\tilde{N}_2 W_2 - N_2 \tilde{W}_2}{N_2 \tilde{W}_1 - \tilde{N}_2 W_1}.
\end{aligned} \tag{B.12}$$

where the N, W and $\tilde{N}$ and $\tilde{W}$ are defined in (3.46). Similarly, from (B.4) and (B.7) and (B.11) we obtain y^i in terms of y^+

$$\begin{aligned} y^i &= \sqrt{2}y^+\frac{\bar{q}^i w_1' - q^i\bar{w}_1'}{\bar{n}_2 w_1' - n_2\bar{w}_1'} + 2\frac{n_2\bar{q}^i - \bar{n}_2 q^i}{\bar{n}_2 w_1' - n_2\bar{w}_1'}, \\ &= \sqrt{2}y^+\frac{Q^i\tilde{W}_1 - P^i W_1}{N_2\tilde{W}_1 - \tilde{N}_2 W_1} + 2\frac{\tilde{N}_1 Q^i - N_2 P^i}{N_2\tilde{W}_1 - \tilde{N}_2 W_1}. \end{aligned} \tag{B.13}$$

We can now substitute the solutions for y^i and y^- in terms of y^+ in the equation (B.5) and obtain the following equation for y^+ after some manipulations

$$\begin{aligned} &\tilde{A}\bar{y}^+y^+ + \tilde{B}(y^+ + \bar{y}^+) + \tilde{C} = 0, \\ &\tilde{A} = -\frac{w_1}{2}\left(\frac{\bar{w}_2' w_1' - w_2'\bar{w}_1'}{\bar{n}_2 w_1' - n_2\bar{w}_1'} - \sum_{i=1}^{8}\frac{(\bar{q}^i w_1' - q^i\bar{w}_1')^2}{(\bar{n}_2 w_1' - n_2\bar{w}_1')^2}\right), \\ &\tilde{B} = -\frac{n_2}{\sqrt{2}}\left(\frac{\bar{w}_2' w_1' - w_2'\bar{w}_1'}{\bar{n}_2 w_1' - n_2\bar{w}_1'} - \sum_{i=1}^{8}\frac{(\bar{q}^i w_1' - q^i\bar{w}_1')^2}{(\bar{n}_2 w_1' - n_2\bar{w}_1')^2}\right), \\ &\tilde{C} = \frac{n_1(\bar{n}_2 w_1' - n_2\bar{w}_1') - n_2(n_2\bar{w}_2' - w_2'\bar{n}_2)}{\bar{n}_2 w_1' - n_2\bar{w}_1'} + n_2\frac{\sum_{i=1}^{8}(n_2\bar{q}^i - \bar{n}_1 q^i)(w_1'\bar{q}^i - q^i\bar{w}_1')}{(\bar{n}_2 w_1' - n_2\bar{w}_1')^2}. \end{aligned} \tag{B.14}$$

Using the above equation and its complex conjugate one can obtain a relationship between y^+ and $\bar{y}^+$ by getting rid of the quadratic term $\bar{y}^+y^+$ in (B.14). This is given by

$$\begin{aligned} &\frac{\bar{y}^+}{\sqrt{2}} = \frac{y^+}{\sqrt{2}} + \\ &\frac{(\bar{n}_2 w_1' - n_2\bar{w}_1')(\bar{w}_1' n_1 - w_1'\bar{n}_1 + n_2\bar{w}_2' - w_2'\bar{n}_2) - \sum_{i=1}^{8}(n_2\bar{q}^i - \bar{n}_1 q^i)(w_1'\bar{q}^i - q^i\bar{w}_1')}{(\bar{n}_2 w_1' - n_2\bar{w}_1')(w_1\bar{w}_2' - w_2'\bar{w}_1') - \sum_{i=1}^{8}(\bar{q}^i w_1' - q^i\bar{w}_1')^2}. \end{aligned} \tag{B.15}$$

Substituting this relationship in (B.14) and writing the equation in terms of only y^+ we obtain the quadratic equation

$$\begin{aligned} &A\left(\frac{y^+}{\sqrt{2}}\right)^2 + B\frac{y^+}{\sqrt{2}} + C = 0, \\ &A = \tilde{W}_2 W_1 - W_2\tilde{W}_1 + \sum_{i=1}^{8}\frac{(Q_i\tilde{W}_1 - P_i W_1)^2}{N_2\tilde{W}_1 - \tilde{N}_2 W_1}, \\ &B = (\tilde{W}_2 N_2 - W_2\tilde{N}_2) - (\tilde{N}_1 W_1 - N_1\tilde{W}_1) + \sum_{i=1}^{8}\frac{(\tilde{N}_2 Q_i - P_i N_2)(\tilde{W}_1 Q_i - P_i W_1)}{N_2\tilde{W}_1 - \tilde{N}_2 W_1}, \\ &C = N_1\tilde{N}_2 - \tilde{N}_1 N_2. \end{aligned} \tag{B.16}$$

Here we have used the definitions (B.2) and (3.46). Note that from (B.12), (B.13) and (B.16) the solution fo the T-moduli $y^+, y^- y^i$ are independent of the axion-dilaton moduli τ and thus solve the attractor equations for the entropy function (3.45). Note that the above procedure of solving the attractor equations for the T-moduli generalize trivially to any model with vector multiplet moduli space

$$\mathcal{M}_V = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)} \tag{B.17}$$

C. Properties of Siegel modular forms

In this section we detail the construction and properties of the Siegel modular forms $\widetilde{\Phi}_0$ and $\widetilde{\Phi}_4$ which are used to write down the dyon partition functions for the two $\mathcal{N}=2$ models discussed in this paper. From and (4.4), (5.3) we see that these modular forms are defined in terms of the modular forms $\widetilde{\Phi}_2$, Φ_6 and $\widetilde{\Phi}_2'$. We first recall the construction and properties of $\widetilde{\Phi}_2$ and $\widetilde{\Phi}_6$ which was constructed to capture the degeneracy of dyons in a $\mathbb{Z}_2$ orbifold of type II theory and the $\mathbb{Z}_2$ CHL orbifold respectively in [10, 7].

$\boldsymbol{\widetilde{\Phi}_6(\tilde{\rho},\tilde{\sigma},\tilde{v})}$. The infinite product representation of Φ_6 is given by [7]

$$\widetilde{\Phi}_6(\tilde{\rho},\tilde{\sigma},\tilde{v}) = -\frac{1}{16}e^{(2\pi i(\frac{\tilde{\sigma}}{2}+\tilde{\rho}+\tilde{v}))}\times \tag{C.1}$$
$$\prod_{r=0}^{1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{2},l,j\in\mathbb{Z}\\ k',l\geq 0, j<0 \text{for } k'=l=0}}\left(1-\exp(2\pi i(k'\tilde{\sigma}+l\tilde{\rho}+j\tilde{v}))\right)^{\sum_{s=0}^{1}(-1)^{sl}c_6^{(r,s)}(4k'l-j^2)},$$

where the coefficients $c_6^{(r,s)}$ are defined by the expansion

$$F_6^{(r,s)}(\tau,z) = \sum_{b\in\mathbb{Z},n} c_6^{(r,s)}(4n-b^2)q^n e^{2\pi i b}, \tag{C.2}$$

here $n\in\mathbb{Z}$ for $r=0$ and $\frac{1}{2}\mathbb{Z}$ for $r=1$. The expressions for various values of (r,s) are as follows: Let

$$F_6^{(r,s)}(\tau,z) = h_{6;0}^{(r,s)}(\tau)\vartheta_3(2\tau,2z) + h_{6;1}^{(r,s)}(\tau)\vartheta_2(2\tau,2z), \tag{C.3}$$

here we list these functions

$$\begin{aligned}
h_{6;0}^{(0,0)}(\tau) &= 8\frac{\vartheta_3(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\frac{1}{\vartheta_3(2\tau,0)},\\
h_{6;1}^{(0,0)}(\tau) &= -8\frac{\vartheta_2(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\frac{1}{\vartheta_2(2\tau,0)},\\
h_{6;0}^{(0,1)}(\tau) &= 2\frac{1}{\vartheta_3(2\tau,0)}, \qquad h_{6;1}^{(0,1)}(\tau) = 2\frac{1}{\vartheta_2(2\tau,0)},\\
h_{6;0}^{(1,0)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_4(\tau,0)^2}, \qquad h_{6;1}^{(1,0)}(\tau) = -4\frac{\vartheta_2(2\tau,0)}{\vartheta_4(\tau,0)^2},\\
h_{6;0}^{(1,1)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_3(\tau,0)^2}, \qquad h_{6;1}^{(1,0)}(\tau) = 4\frac{\vartheta_2(2\tau,0)}{\vartheta_3(\tau,0)^2}.
\end{aligned} \tag{C.4}$$

We can now define the coefficients $c^{(r,s)}(u)$ through the expansions

$$h_{6;0}^{(r,s)}(\tau) = \sum_n c_6^{(r,s)}(4n)q^n, \qquad h_{6;1}^{(r,s)}(\tau) = \sum_n c_6^{(r,s)}(4n)q^n. \tag{C.5}$$

From (C.4) we see that in the expansion of $h_{6;l}^{(r,s)}$, $n\in\mathbb{Z}-\frac{l}{4}$ for $r=0$ and $n\in\frac{1}{2}\mathbb{Z}-\frac{l}{4}$ for $r=1$. Thus for given (r,s) the $c^{(r,s)}(u)$ defined through the two equations in (C.5) have non-overlapping set of arguments. Substituting (C.5) in (C.3) we get (C.2) The properties of $\widetilde{\Phi}_6$ can be studied by obtaining it as a result of a threshold like integral [7]. Here we review some of its important properties:

1. $\widetilde{\Phi}_6$ is a modular form of weight 6 under the subgroup $\tilde{G}$ of $\mathrm{Sp}(2,\mathbb{Z})$ defined in [6]. Therefore

$$\widetilde{\Phi}_6((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^6\tilde{\Phi}_6(\Omega), \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{G}.$$
$$\Omega = \begin{pmatrix} \tilde{\rho} & \tilde{v} \\ \tilde{v} & \tilde{\rho} \end{pmatrix} \tag{C.6}$$

$\tilde{G}$ contains the subgroup $\tilde{H}$ [6] whose elements are of the form:

$$\begin{pmatrix} a & -b & b & 0 \\ -c & d & 0 & c \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix}, \quad ad-bc=1, c \in 2\mathbb{Z}. \tag{C.7}$$

From (C.6) we see that $\widetilde{\Phi}_6$ is invariant under the $\mathrm{Sp}(2,\mathbb{Z})$ transformation given in (C.7). This gives that for

$$\begin{aligned} \tilde{\rho}' &= a^2\tilde{\rho} + b^2\tilde{\sigma} - 2ab\tilde{v} + ab, \\ \tilde{\sigma}' &= c^2\tilde{\rho} + d^2\tilde{\sigma} - 2cd\tilde{v} + cd, \\ \tilde{v}' &= -ac\tilde{\rho} - bc\tilde{\sigma} + (ad+bc)\tilde{v} - bc, \end{aligned} \tag{C.8}$$

we have

$$\widetilde{\Phi}_6(\rho', \sigma' v') = \Phi_6(\rho, \sigma, v) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc=1, c\in 2\mathbb{Z}. \tag{C.9}$$

Thus these group of matrices belong to $\Gamma_0(2)$.

2. From examining the coefficients $c_6^{(r,s)}$ defined by the expansions in (C.4) it can be seen that

$$c_6^{(r,0)}(u) \pm c_6^{r,1}(u) \in 2\mathbb{Z} \tag{C.10}$$

3. $\widetilde{\Phi}_6$ has second order zeros at

$$\begin{aligned} \big(n_2(\tilde{\sigma}\tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho}m_1 + m_2\big) &= 0, \\ \text{for } m_1 \in 2\mathbb{Z}, m_2, n_2 \in \mathbb{Z}, b \in 2\mathbb{Z}+1, m_1n_1 + m_2n_2 + \frac{b^2}{4} &= \frac{1}{4}. \end{aligned} \tag{C.11}$$

4. In the limit $\tilde{v} \to 0$, $\widetilde{\Phi}_6$ factorizes as

$$\widetilde{\Phi}_6(\tilde{\rho}, \tilde{\sigma}, \tilde{v})_{\tilde{v}\to 0} = -\frac{1}{4}\pi^2\tilde{v}^2\eta(\tilde{\rho})^8\eta(2\tilde{\rho})^8\eta\Big(\frac{\tilde{\sigma}}{2}\Big)^8\eta(\tilde{\sigma})^8. \tag{C.12}$$

5. Under the $\mathrm{Sp}(2,\mathbb{Z})$ transformation given in (4.17) $\tilde{\Phi}_6$ is related to the $\mathrm{Sp}(2,\mathbb{Z})$ modular form of weight 6 by

$$\begin{aligned} \widetilde{\Phi}_6(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) &= \tilde{\sigma}^{-6}\Phi_6\left(\tilde{\rho} - \frac{\tilde{v}^2}{\tilde{\sigma}}, \frac{\tilde{\rho}\tilde{\sigma} - (\tilde{v}-1)^2}{\tilde{\sigma}}, \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}}{\tilde{\sigma}}\right), \\ &= \tilde{\sigma}^{-6}\Phi_6(\rho, \sigma, v), \end{aligned} \tag{C.13}$$

where Φ_6 is defined by

$$\Phi_6(\rho,\sigma,v) = -\exp(2\pi i(\rho+\sigma+v)) \prod_{r,s=0}^{1} \prod_{\substack{(k,l,b)\in\mathbb{Z} \\ k,l\geq 0, b<0 \text{ for } k=l=0}} \{1-(-1)^r \exp(2\pi i(k\sigma+l\rho+bv))\}^{c_6^{(r,s)}(4kl-b^2)}, \tag{C.14}$$

$(\tilde{\rho},\tilde{\sigma},\tilde{v})$ is related (ρ,σ,v) by (4.17) Φ_6 has the factorization property in the limit $v\to 0$

$$\Phi_6(\rho,\sigma,v)_{v\to 0} = 4\pi^2 v^2 \eta(\sigma)^8\eta(2\sigma)^8\eta(\rho)^8\eta(2\rho)^8. \tag{C.15}$$

6. Using (C.13) and the factorization property (C.15) we see that when $\tilde{\rho}\sigma-\tilde{v}^2+\tilde{v}\sim 0$ $\tilde{\Phi}_6$ factorizes as

$$\tilde{\Phi}_6(\tilde{\rho},\tilde{\sigma},\tilde{v}) \sim 4\pi^2(2v-\rho-\sigma)^2 v^2\eta(\sigma)^8\eta(2\sigma)^8\eta(\rho)^8\eta(2\rho)^8. \tag{C.16}$$

$\mathbf{\widetilde{\Phi}_2(\tilde{\rho},\tilde{\sigma},\tilde{v})}$**.** The infinite product representation of $\widetilde{\Phi}_2$ is given by [10]

$$\widetilde{\Phi}_2(\tilde{\rho},\tilde{\sigma},\tilde{v}) = -\frac{1}{2^8}e^{(2\pi i(\tilde{\rho}+\tilde{v}))}\times \prod_{r=0}^{1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{2},l,j\in\mathbb{Z} \\ k',l\geq 0, j<0 \text{for } k'=l=0}} \left(1-\exp(2\pi i(k'\tilde{\sigma}+l\tilde{\rho}+j\tilde{v}))\right)^{\sum_{s=0}^{1}(-1)^{sl}c_2^{(r,s)}(4k'l-j^2)}, \tag{C.17}$$

where the coefficients $c_2^{(r,s)}$ are defined by the expansion

$$F_2^{(r,s)}(\tau,z) = \sum_{b\in\mathbb{Z},n} c_2^{(r,s)}(4n-b^2)q^n e^{2\pi i b}, \tag{C.18}$$

here $n\in\mathbb{Z}$ for $r=0$ and $\frac{1}{2}\mathbb{Z}$ for $r=1$. The expressions for various values of (r,s) are as follows: Let

$$F_2^{(r,s)}(\tau,z) = h_{2;0}^{(r,s)}(\tau)\vartheta_3(2\tau,2z) + h_{2;1}^{(r,s)}(\tau),\vartheta_2(2\tau,2z). \tag{C.19}$$

here we list these functions

$$\begin{aligned}
h_{2;0}^{(0,0)}(\tau) &= 0, & h_{2;1}^{(0,0)}(\tau) &= 0, \\
h_{2;0}^{(0,1)}(\tau) &= 4\frac{1}{\vartheta_3(2\tau,0)}, & h_{2;1}^{(0,1)}(\tau) &= 4\frac{1}{\vartheta_2(2\tau,0)}, \\
h_{2;0}^{(1,0)}(\tau) &= 8\frac{\vartheta_3(2\tau,0)}{\vartheta_4(\tau,0)^2}, & h_{2;1}^{(1,0)}(\tau) &= -8\frac{\vartheta_2(2\tau,0)}{\vartheta_4(\tau,0)^2}, \\
h_{2;0}^{(1,1)}(\tau) &= 8\frac{\vartheta_3(2\tau,0)}{\vartheta_3(\tau,0)^2}, & h_{6;1}^{(1,0)}(\tau) &= 8\frac{\vartheta_2(2\tau,0)}{\vartheta_3(\tau,0)^2}.
\end{aligned} \tag{C.20}$$

We can now define the coefficients $c^{(r,s)}(u)$ through the expansions

$$h_{2;0}^{(r,s)}(\tau) = \sum_n c_2^{(r,s)}(4n)q^n, \qquad h_{2;1}^{(r,s)}(\tau) = \sum_n c_2^{(r,s)}(4n)q^n. \tag{C.21}$$

From (C.20) we see that in the expansion of $h_{2;l}^{(r,s)}$, $n \in \mathbb{Z} - \frac{l}{4}$ for $r = 0$ and $n \in \frac{1}{2}\mathbb{Z} - \frac{l}{4}$ for $r = 1$. Thus for given (r,s) the $c^{(r,s)}(u)$ defined through the two equations in (C.21) have non-overlapping set of arguments. Substituting (C.21) in (C.19) we get (C.18) The properties of $\widetilde{\Phi}_2$ can be studied by obtaining it as a result of a threshold like integral [10]. Here we review some of its important properties:

1. $\widetilde{\Phi}_2$ is a modular form of weight 2 under the subgroup $\tilde{G}$ of $\mathrm{Sp}(2,\mathbb{Z})$ defined in [6]. Therefore

$$\widetilde{\Phi}_2((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^2 \tilde{\Phi}_2(\Omega), \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{G}. \tag{C.22}$$

   Since the $\tilde{G}$ contains $\tilde{H}$ given in (C.7) using the same arguments as in the case of $\widetilde{\Phi}_6$ we see that $\widetilde{\Phi}_2$ is also invariant under the transformation (C.8).

2. From examining the coefficients $c_2^{(r,s)}$ defined by the expansions in (C.20) it can be seen that

$$c_2^{(r,0)}(u) \pm c_2^{r,1}(u) \in 2\mathbb{Z}. \tag{C.23}$$

3. $\widetilde{\Phi}_2$ has second order zeros at

$$\begin{aligned} \big(n_2(\tilde{\sigma}\tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho}m_1 + m_2\big) &= 0, \qquad \text{(C.24)} \\ \text{for } m_1 \in 2\mathbb{Z}, m_2, n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z} b \in 2\mathbb{Z}+1, m_1n_1 + m_2n_2 + \frac{b^2}{4} &= \frac{1}{4}. \end{aligned}$$

   It also has a second order pole at

$$\begin{aligned} \big(n_2(\tilde{\sigma}\tilde{\rho} - \tilde{v}^2) + b\tilde{v} + n_1\tilde{\sigma} - \tilde{\rho}m_1 + m_2\big) &= 0, \text{(C.25)} \\ \text{for } m_1 \in 2\mathbb{Z}+1, m_2, n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z}, b \in 2\mathbb{Z}+1, m_1n_1 + m_2n_2 + \frac{b^2}{4} &= \frac{1}{4}. \end{aligned}$$

4. In the limit $\tilde{v} \to 0$, $\widetilde{\Phi}_2$ factorizes as

$$\widetilde{\Phi}_2(\tilde{\rho}, \tilde{\sigma}, \tilde{v})_{\tilde{v}\to 0} = -\frac{\pi^2}{64}\tilde{v}^2 \frac{\eta(2\tilde{\rho})^{16}}{\eta(\tilde{\rho})^8} \frac{\eta(\tilde{\sigma}/2)^{16}}{\eta(\tilde{\sigma})^8}. \tag{C.26}$$

5. Under the $\mathrm{Sp}(2,\mathbb{Z})$ transformation given in (4.17) $\tilde{\Phi}_2$ is related to Φ_2 $\mathrm{Sp}(2,\mathbb{Z})$ modular form of weight 2 by

$$\begin{aligned} \widetilde{\Phi}_2(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) &= \tilde{\sigma}^{-2}\Phi_2\left(\tilde{\rho} - \frac{\tilde{v}^2}{\tilde{\sigma}}, \frac{\tilde{\rho}\tilde{\sigma} - (\tilde{v}-1)^2}{\tilde{\sigma}}, \frac{\tilde{\rho}\tilde{\sigma} - \tilde{v}^2 + \tilde{v}}{\tilde{\sigma}}\right), \\ &= \tilde{\sigma}^{-2}\Phi_2(\rho, \sigma, v). \end{aligned} \tag{C.27}$$

   where Φ_2 is defined by

$$\begin{aligned} \Phi_2(\rho, \sigma, v) &= -\exp(2\pi i(\rho + \sigma + v)) \\ &\prod_{r,s=0}^{1} \prod_{\substack{(k,l,b)\in\mathbb{Z} \\ k,l\geq 0, b<0 \text{ for } k=l=0}} \{1 - (-1)^r \exp(2\pi i(k\sigma + l\rho + bv))\}^{c_2^{(r,s)}(4kl-b^2)}, \end{aligned} \tag{C.28}$$

$(\tilde\rho,\tilde\sigma,\tilde v)$ is related (ρ,σ,v) by (4.17) Φ_2 has the factorization property in the limit $v \to 0$

$$\Phi_2(\rho,\sigma,v)_{v\to 0} = 4\pi^2 v^2 \frac{\eta(2\rho)^{16}}{\eta(\rho)^8}\frac{\eta(2\sigma)^{16}}{\eta(\sigma)^8}. \tag{C.29}$$

6. Using (C.27) and the factorization property (C.29) we see that when $\tilde\rho\sigma - \tilde v^2 + \tilde v \sim 0$ $\tilde\Phi_2$ factorizes as

$$\tilde\Phi_2(\tilde\rho,\tilde\sigma,\tilde v) \sim 4\pi^2(2v-\rho-\sigma)^2 v^2 \frac{\eta(2\rho)^{16}}{\eta(\rho)^8}\frac{\eta(2\sigma)^{16}}{\eta(\sigma)^8}. \tag{C.30}$$

$\widetilde\Phi_2'(\tilde\rho,\tilde\sigma,\tilde v)$. $\widetilde\Phi_2'$ is a $\mathrm{Sp}(2,\mathbb{Z})$ modular form of weight 2 which is related to $\widetilde\Phi_2$ by the following

$$\widetilde\Phi_2'(\tilde\rho,\tilde\sigma,\tilde v) = \widetilde\Phi_2(\frac{\tilde\sigma}{2}, 2\tilde\rho, \tilde v), \tag{C.31}$$

Though we can derive many of the properties of $\widetilde\Phi_2'$ from the properties of $\widetilde\Phi_2$ it is instructive to write down the threshold integral from which $\widetilde\Phi_2'$ can be obtained This threshold integral is given by

$$\mathcal{I}(\tilde\rho,\tilde\sigma,\tilde v) = \sum_{l,r,s=0}^{1} \mathcal{I}_{r,s,l}, \tag{C.32}$$

where

$$\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_2,n_2\in\mathbb{Z}, n_1\in\mathbb{Z}/2\\ m_1\in 2\mathbb{Z}+r, b\in 2\mathbb{Z}+l}} q^{p_L^2/2}\bar q^{p_R^2/2} e^{(2\pi i n_1 s)} h_{2;l}^{(r,s)}, \tag{C.33}$$

where $\mathcal{F}$ denotes the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ in the upper half plane, $h_{2;l}^{(r,s)}$ are given in (C.20). Here

$$\begin{aligned} q &= e^{2\pi i\tau}, & \text{(C.34)}\\ \frac{p_R^2}{2} &= \frac{1}{4\det \mathrm{Im}\Omega}|-m_1\tilde\rho + m_2 + n_1\tilde\sigma + n_2(\tilde\sigma\tilde\rho - \tilde v^2) + b\tilde v|^2, & \text{(C.35)}\\ \frac{p_L^2}{2} &= \frac{1}{2}p_R^2 + m_1 n_1 + m_2 n_2 + \frac{1}{4}b^2, & \text{(C.36)}\\ \Omega &= \begin{pmatrix} \tilde\rho & \tilde v \\ \tilde v & \tilde\sigma \end{pmatrix}. & \text{(C.37)} \end{aligned}$$

The integrals in (C.32) can be performed using the procedure [35–37, 7]. The procedure involves evaluating the contribution of the zero orbit, the degenerate orbit and the non-degenerate orbit of $\mathrm{SL}(2,\mathbb{Z})$ separately. The result is

$$\mathcal{I} = -2\ln\left[2^{16}\kappa(\det \mathrm{Im}\,\Omega)^2\right] - 2\ln \widetilde\Phi_2'(\tilde\rho,\tilde\sigma,\tilde v) - 2\ln \bar{\widetilde\Phi}_2'(\tilde\rho,\tilde\sigma,\tilde v), \tag{C.38}$$

where

$$\kappa = \left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}\right)^4. \tag{C.39}$$

γ_E is Euler's constant and $\widetilde{\Phi}_2'$ is given by

$$\widetilde{\Phi}_2'(\tilde{\rho},\tilde{\sigma},\tilde{v}) = -\frac{1}{2^8} e^{(2\pi i(\frac{\tilde{\sigma}}{2}+\tilde{v}))} \times \tag{C.40}$$
$$\prod_{r=0}^{1} \prod_{\substack{k'\in 2\mathbb{Z}+r, l\in\mathbb{Z}/2, j\in\mathbb{Z} \\ k',l\geq 0, j<0 \,\mathrm{for}\, k'=l=0}} \big(1-\exp(2\pi i(k'\tilde{\sigma}+l\tilde{\rho}+j\tilde{v})\big)^{\sum_{s=0}^{1}\exp(2\pi i s l) c_2^{(r,s)}(4k'l-j^2)},$$

From the threshold integral in (C.38) we can obtain the following properties of $\widetilde{\Phi}_2'$.

1. From (C.33) it is easy to see that those $SO(2,3;\mathbb{Z}) = Sp(2,\mathbb{Z})$ transformation which, acting on the vector (m_1,n_2,n_1,n_2,b) with $m_1 m_2, n_2, b$ integers and $n_1 \in \mathbb{Z}/2$ preserves m_1 modulo 2, $n_1,, m_2, n_2$ modulo 1 and b modulo 2, will be symmetries of $\mathcal{I}$ in (C.32). This defines the subgroup $\tilde{G}$ [7], thus from (C.38) we see that $\widetilde{\Phi}_2'$ is a modular form of weight 2 under the subgroup $\tilde{G}$ of $Sp(2,\mathbb{Z})$. Therefore

$$\widetilde{\Phi}_2'((A\Omega+B)(C\Omega+D)^{-1}) = \det(C\Omega+D)^2\widetilde{\Phi}_2'(\Omega), \qquad \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \tilde{G}. \tag{C.41}$$

   Since the $\tilde{G}$ contains $\tilde{H}$ given in (C.7) using the same arguments as in the case of $\widetilde{\Phi}_6$ we see that $\widetilde{\Phi}_2'$ is also invariant under the transformation (C.8).

2. $\widetilde{\Phi}_2'$ has second order zeros at

$$\big(n_2(\tilde{\sigma}\tilde{\rho}-\tilde{v}^2)+b\tilde{v}+n_1\tilde{\sigma}-\tilde{\rho}m_1+m_2\big) = 0, \tag{C.42}$$
$$\mathrm{for}\, m_1\in 2\mathbb{Z}, m_2,n_2\in\mathbb{Z}, n_1\in\mathbb{Z} b\in 2\mathbb{Z}+1, m_1n_1+m_2n_2+\frac{b^2}{4} = \frac{1}{4}.$$

   It also has a second order pole at

$$\big(n_2(\tilde{\sigma}\tilde{\rho}-\tilde{v}^2)+b\tilde{v}+n_1\tilde{\sigma}-\tilde{\rho}m_1+m_2\big) = 0, \tag{C.43}$$
$$\mathrm{for}\, m_1\in 2\mathbb{Z}, m_2,n_2\in\mathbb{Z}, n_1\in\mathbb{Z}+\frac{1}{2}, b\in 2\mathbb{Z}+1, m_1n_1+m_2n_2+\frac{b^2}{4} = \frac{1}{4}.$$

3. In the limit $\tilde{v}\to 0$, $\widetilde{\Phi}_2'$ factorizes as

$$\widetilde{\Phi}_2'(\tilde{\rho},\tilde{\sigma},\tilde{v})_{\tilde{v}\to 0} = -\frac{\pi^2}{64}\tilde{v}^2\frac{\eta(\tilde{\rho})^{16}}{\eta(2\tilde{\rho})^8}\frac{\eta(\tilde{\sigma})^{16}}{\eta(\tilde{\sigma}/2)^8}. \tag{C.44}$$

4. Under the $Sp(2,\mathbb{Z})$ transformation given in (4.17) from the relation of $\widetilde{\Phi}_2$ to $\widetilde{\Phi}_2'$ in (C.31) it can be shown that $\tilde{\Phi}_2'$ is related to $\hat{\Phi}_2$ $Sp(2,\mathbb{Z})$ modular form of weight 2 by

$$\begin{aligned}\widetilde{\Phi}_2'(\tilde{\rho},\tilde{\sigma},\tilde{v}) &= \tilde{\sigma}^{-2}\hat{\Phi}_2'\left(\tilde{\rho}-\frac{\tilde{v}^2}{\tilde{\sigma}}, \frac{\tilde{\rho}\tilde{\sigma}-(\tilde{v}-1)^2}{\tilde{\sigma}}, \frac{\tilde{\rho}\tilde{\sigma}-\tilde{v}^2+\tilde{v}}{\tilde{\sigma}}\right), \\ &= \tilde{\sigma}^{-2}\hat{\Phi}_2'(\rho,\sigma,v).\end{aligned} \tag{C.45}$$

where $\hat{\Phi}_2$ is defined by [11].

$$\begin{aligned}\hat{\Phi}_2(\rho,\sigma,v) &= -\exp(2\pi i(\rho+\sigma+v)) && \text{(C.46)}\\ &\prod_{b=0}^{1}\prod_{r,s=0}^{1}\prod_{\substack{(k,l)\in\mathbb{Z},j\in 2\mathbb{Z}+b\\ k,l\geq 0,j<0\,\text{for}\,k=l=0}}\{1-(-1)^r\exp(2\pi i(k\sigma+l\rho+jv))\}^{\hat{c}^{(r,s)}_{2;b}(4kl-j^2)},\end{aligned}$$

$(\tilde{\rho},\tilde{\sigma},\tilde{v})$ is related (ρ,σ,v) by (4.17). $\hat{c}^{(r,s)}_{2;b}$ is related to $c^{(r,s)}_2$ by

$$\hat{c}^{(r,s)}_{2;b}(u)=\frac{1}{2}\sum_{r'=0}^{1}\sum_{s'=0}^{1}e^{2\pi i(sr'-rs')/2)}c^{(r',s')}_{2;b}(u). \qquad \text{(C.47)}$$

$\hat{\Phi}_2$ has the factorization property in the limit $v\to 0$

$$\hat{\Phi}_2(\rho,\sigma,v)_{v\to 0}=4\pi^2v^2\frac{\eta(\rho)^{16}}{\eta(2\rho)^8}\frac{\eta(\sigma)^{16}}{\eta(2\sigma)^8}. \qquad \text{(C.48)}$$

5. Using (C.27) and the factorization property (C.29) we see that when $\tilde{\rho}\sigma-\tilde{v}^2+\tilde{v}\sim 0$ $\tilde{\Phi}'_2$ factorizes as

$$\tilde{\Phi}'_2(\tilde{\rho},\tilde{\sigma},\tilde{v})\sim 4\pi^2(2v-\rho-\sigma)^2v^2\frac{\eta(\rho)^{16}}{\eta(2\rho)^8}\frac{\eta(\sigma)^{16}}{\eta(2\sigma)^8}. \qquad \text{(C.49)}$$

Now that we have the properties of the basic modular forms $\widetilde{\Phi}_2,\widetilde{\Phi}'_2,\widetilde{\Phi}_6$ we can derive the properties of the modular forms $\widetilde{\Phi}'_0$ and $\widetilde{\Phi}'_4$ which are used in the writing down the dyon partition functions in the STU model and the FHSV model.

$\widetilde{\Phi}_0(\tilde{\rho},\tilde{\sigma},\tilde{v})$. We define $\widetilde{\Phi}_0$ as

$$\widetilde{\Phi}_0=\widetilde{\Phi}_2\sqrt{\frac{\widetilde{\Phi}'_2}{\widetilde{\Phi}_6}}. \qquad \text{(C.50)}$$

1. It is clear that from the above definition this is a modular form of weight 0 under the subgroup $\tilde{G}$. Since $\tilde{G}$ contains $\tilde{H}$ it is invariant under the transformations (C.8).

2. From (C.11), (C.24) and (C.42) and from the definition (C.50) we see that $\widetilde{\Phi}_0$ has second order zeros at

$$\big(n_2(\tilde{\sigma}\tilde{\rho}-\tilde{v}^2)+b\tilde{v}+n_1\tilde{\sigma}-\tilde{\rho}m_1+m_2\big)=0, \qquad \text{(C.51)}$$
$$\text{for}\, m_1\in 2\mathbb{Z},m_2,n_2\in\mathbb{Z},n_1\in\mathbb{Z}b\in 2\mathbb{Z}+1,m_1n_1+m_2n_2+\frac{b^2}{4}=\frac{1}{4}.$$

Further more from the form the locations of the poles of $\widetilde{\Phi}_2$ given in (C.25) and $\widetilde{\Phi}'_2$ in (C.43) and from (C.50) we see that $\widetilde{\Phi}_0$ has second order poles at

$$\big(n_2(\tilde{\sigma}\tilde{\rho}-\tilde{v}^2)+b\tilde{v}+n_1\tilde{\sigma}-\tilde{\rho}m_1+m_2\big)=0, \qquad \text{(C.52)}$$
$$\text{for}\, m_1\in 2\mathbb{Z}+1,m_2,n_2\in\mathbb{Z},n_1\in\mathbb{Z},b\in 2\mathbb{Z}+1,m_1n_1+m_2n_2+\frac{b^2}{4}=\frac{1}{4}.$$

It has simple poles at

$$\big(n_2(\tilde\sigma\tilde\rho - \tilde v^2) + b\tilde v + n_1\tilde\sigma - \tilde\rho m_1 + m_2\big) = 0, \quad \text{(C.53)}$$

$$\text{for } m_1 \in 2\mathbb{Z}, m_2, n_2 \in \mathbb{Z}, n_1 \in \mathbb{Z} + \frac{1}{2}, b \in 2\mathbb{Z}+1, m_1 n_1 + m_2 n_2 + \frac{b^2}{4} = \frac{1}{4}.$$

3. From the factorization properties (C.12) and (C.26) and (C.44) and from the definition (C.50) we see that in the limit $\tilde v \to 0$ $\widetilde{\Phi}_0$ factorizes as

$$\widetilde{\Phi}_0(\tilde\rho,\tilde\sigma,\tilde v)_{\tilde v\to 0} = -\frac{\pi^2}{256}\tilde v^2 \frac{\eta(2\tilde\rho)^8}{\eta(\tilde\rho)^4}\frac{\eta(\tilde\sigma/2)^8}{\eta(\tilde\sigma)^4}. \qquad \text{(C.54)}$$

4. (C.10) and (C.23) ensure that in the the product formula for $\widetilde{\Phi}_6$, Φ_2 and Φ_2' given in (C.1), (C.17) and (C.40) the exponents $c_6^{(r,0)}(u) \pm c_6^{(r,1)}$ and $c_2^{(r,0)}(u) \pm c_2^{(r,1)}(u)$ are all even integers. The square roots involved in obtaining $\widetilde{\Phi}_0$ just make these combinations integers. Therefore we have

$$\frac{K}{\widetilde{\Phi}_0(\tilde\rho,\tilde\sigma,\tilde v)} = \sum_{\substack{m,n,p \\ m\geq -1/2, n\geq 1/2}} e^{2\pi i(m\tilde\rho + n\tilde\sigma + p\tilde v)} g(m,n,p), \qquad \text{(C.55)}$$

where the Fourier coefficients $g(m,n,p)$ are integers with $K = -2^{-10}$. $m \in \mathbb{Z}/2$, $n \in \mathbb{Z}/2$ and $j \in \mathbb{Z}$. The lower bound in the sum in (C.55) and the domains of m, n, p are obtained by examining the product representation (C.1), (C.17) and (C.40).

5. Using (C.13), (C.27) and (C.45) it is easily seen that under the $\mathrm{Sp}(2,\mathbb{Z})$ transformation given in (4.17) $\widetilde{\Phi}_0$ is related to Φ_0 the $\mathrm{Sp}(2,\mathbb{Z})$ modular form of weight 2 by

$$\begin{aligned}\widetilde{\Phi}_0(\tilde\rho,\tilde\sigma,\tilde v) &= \Phi_0\left(\tilde\rho - \frac{\tilde v^2}{\tilde\sigma}, \frac{\tilde\rho\tilde\sigma - (\tilde v - 1)^2}{\tilde\sigma}, \frac{\tilde\rho\tilde\sigma - \tilde v^2 + \tilde v}{\tilde\sigma}\right), \\ &= \Phi_0(\rho,\sigma,v).\end{aligned} \qquad \text{(C.56)}$$

where Φ_0 is defined by

$$\Phi_0(\rho,\sigma,v) = \Phi_2(\rho,\sigma,v)\sqrt{\frac{\hat\Phi_2(\rho,\sigma,v)}{\Phi_6(\rho,\sigma,v)}}. \qquad \text{(C.57)}$$

$(\tilde\rho,\tilde\sigma,\tilde v)$ is related (ρ,σ,v) by (4.17). From (C.15) and (C.29) and (C.48) it is seen that Φ_0 has the factorization property in the limit $v \to 0$

$$\begin{aligned}\Phi_0(\rho,\sigma,v)_{v\to 0} &= 4\pi^2 v^2 \frac{\eta(2\rho)^8}{\eta(\rho)^4}\frac{\eta(2\sigma)^8}{\eta(\sigma)^4}, \\ &= 4\pi v^2 \vartheta_2(\rho)^4 \vartheta_2(\sigma)^4.\end{aligned} \qquad \text{(C.58)}$$

In the last line of the above equation we have written the Dedekind-η functions in terms of the Jacobi-ϑ functions.

6. Using (C.56) and the factorization property (C.58) we see that when $\tilde\rho\sigma - \tilde v^2 + \tilde v \sim 0$ $\tilde\Phi_0$ factorizes as

$$\tilde\Phi_0(\tilde\rho,\tilde\sigma,\tilde v) \sim 4\pi^2(2v - \rho - \sigma)^2 v^2 \vartheta_2(\rho)^4 \vartheta_2(\sigma)^4. \qquad \text{(C.59)}$$

$\widetilde{\Phi}_4(\tilde{\rho}, \tilde{\sigma}, \tilde{v})$. We will not go into the details of the derivation of the properties of $\widetilde{\Phi}_4$, but it is now clear that from the properties of the form $\widetilde{\Phi}_6$ and $\widetilde{\Phi}_2$ and using the definition

$$\widetilde{\Phi}_4(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = \sqrt{\Phi_6(\tilde{\rho}, \tilde{\sigma}, \tilde{v})\Phi_2(\tilde{\rho}, \tilde{\sigma}, \tilde{v})}, \tag{C.60}$$

one can show the properties listed in section 5. of the paper.

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